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Concept of modeling uncertainly defined shape of the boundary in two-dimensional boundary value problems and verification of its reliability

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ABSTRACT

The paper presents the concept of modeling uncertainly defined shape of the boundary in two-dimensional boundary value problems described by Laplace equation. The authors propose generalization of parametric integral equations system (PIES) in modeling and solving the problems. Previously, PIES was successfully applied to model precisely defined problems. Pseudo-spectral method and modified interval arithmetic are used to define the boundary of the problem and to obtain numerical solution of generalized PIES. We examine direct application of interval Bézier curves in PIES and then we propose our new concept of modeling of the shape of the boundary to improve the reliability of solutions.

1. Introduction

Modelling of the boundary value problems in classical way means that the shape of the boundary and boundary conditions are precisely defined. However, that approach idealizes reality, because it does not consider the measurement errors. The theory of measurement is clearly described in [1]. Furthermore, it is known, that even mathematical definition (by differential equations) of various physical phenomena is too idealized and does not consider all features of the modeled phenomenon. Therefore, differential equations, as well as the shape of the boundary and boundary conditions, do not accurately model physical phenomenon. Nonetheless, new methods for solving such problems are still being developed. The most popular techniques of numerical solution of such problems are based on the discretization of the domain (or boundary) into a number of smaller segments (elements). Each of elements is described by interpolation function and after complex numerical calculations, solutions of the problem are obtained.

Above mentioned approach is used in finite element method (FEM) [2] and boundary element method (BEM) [3]. The main problem of the strategy is to enforce continuity of interpolating functions at the elements connection points . Additionally, in case of BEM, continuity condition of the shape of the boundary must be guaranteed. Nevertheless, the uncertain definition of the problem is not contemplated in FEM and BEM methods. So it is impossible to use the methods directly to solve uncertainly defined problems. This is due to the fact, that there is no possibility to define the problems using traditional mathematical formalism.

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In recent years, many researchers in various fields have increased their focus on application of popular models of imprecision and uncertainty [4,5]. For example: interval numbers and interval arithmetic [6] or theory of fuzzy sets [7]. These are used in well-known techniques of modeling and solving the boundary value problems. Consequently, interval finite element method [8] and interval boundary element method [9] have been developed. However, they inherit several disadvantages from precise version of the above methods. This detriments strongly influence the effectiveness of their application. It will be very problematic to consider the uncertainly defined shape of the boundary in modeling process, due to the discretization of the domain or the boundary. Additionally, in case of classical boundary integral equations (BIE) there is a need to meet the Lyapunov condition for the shape of the boundary [10] or to enforce the continuity condition at the elements connection points. These difficulties occur in problems defined in a precise way. Therefore, in case of uncertain modeling, this will be even more troublesome.

Contrary to the traditional boundary integral equation (BIE), in PIES the approximation of the shape of the boundary is separated from the approximation of the boundary functions. Consequently, to solve uncertainly defined problems, we propose generalization of parametric integral equations system (PIES). PIES was successfully applied for solving problems modeled in a precise way [11–13]. PIES was obtain as analytical modification of the traditional BIE. The shape of the boundary in PIES is modeled using curves well-known in computer graphics [33]. They are directly included into BIE, during its analytical modification. It gives a possibility of creating various continuous shapes of boundary using control points only. It means, that PIES automatically adapts to modifications of shapes of the boundaries. Such advantage significantly simplified solving of the boundary value problems and should be even more noticeable during modeling of uncertainly defined shape of the boundary.

In the literature connected with FEM and BEM, many authors focus only on the interval definition of the boundary conditions and of some parameters. The problem of uncertainly define shape of the boundary is often reduced, for example, to interval length of beam for specific boundary value problem [14]. Nevertheless, the uncertainly defined shape of the boundary for two- and three-dimensional problems are not considered in the literature. So in this paper we generally focus on the shape of the boundary defined by curves (such as Bézier or B-spline) with interval (uncertain) control points. Such way of modeling allows to obtain any shape, not only the beam. However, it is definitely more complicated than uncertainly defined boundary conditions or parameters, which are mentioned in the literature.

Therefore, we decide to generalize PIES for solving the boundary value problems with uncertainly defined shape of the boundary. For this purpose we use interval numbers and interval arithmetic. Nonetheless, direct application of the classical interval arithmetic (without detailed analysis of all arithmetic calculations in every stage of PIES algorithm) results in a very wide interval solutions. Such problem is caused by the overestimation. The way of controlling the overestimation is described in [4], but we decide to consider this problem from more practical point of view. So we decide to verify the reliability of application of interval arithmetic (for modeling of uncertainty) on three stages: modeling of uncertainly defined shape, obtaining interval system of equations in generalized PIES and solving this system.

2. The way of solving uncertainly defined boundary value problems

Many authors applied BEM [15,16], FEM [17,18], finite difference method (FDM) [19,20] or meshless methods [21,22] in numerical modeling and solving boundary value problems. In above mentioned methods, amongst others the interval arithmetic is used to define the uncertainty. These methods are very popular, however, direct application of them, to solve uncertainly defined problems, is not possible. The studies of mentioned methods for uncertainly defined problems [8,9] mainly focused on the uncertainty of boundary conditions or properties of the medium. Uncertain definition of the shape of the boundary in FEM or BEM is really troublesome. The main problem is related to discretization of the domain in FEM or of the boundary in BEM. Therefore, in this paper we decided to generalize PIES to modeling and solving uncertainly defined problems, because it does not require traditional discretization. The generalization of PIES is focus mainly on uncertainly defined shape of the boundary. For modeling of uncertainty we use classical and directed interval arithmetic.We also propose a modification of directed arithmetic to obtain more reliable solutions.

2.1. Basic concepts of interval arithmetic

In PIES for modeling of boundary value problems with uncertainly defined shape of the boundary, we use interval curves. We apply interval numbers to describe such problems, and interval arithmetic [6] to solve them.

Classical interval number $\mathbf{x}$ (proper interval or shorter interval) is the set of real numbers x which meet the condition [6] $\mathbf{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} \mid \underline{x} \le x \le \overline{x}\}$, where $\underline{x}$ - is infimum and $\overline{x}$ - is supremum of interval $\mathbf{x}$. Additionally, to operate on these numbers, special interval arithmetic was created and it is generally defined by the formula [6]:

$$\mathbf{x} \circ \mathbf{y} = [\underline{x}, \overline{x}] \circ [\underline{y}, \overline{y}] = [min(\underline{x} \circ \underline{y}, \underline{x} \circ \overline{y}, \overline{x} \circ \underline{y}, \overline{x} \circ \overline{y}), max(\underline{x} \circ \underline{y}, \underline{x} \circ \overline{y}, \overline{x} \circ \underline{y}, \overline{x} \circ \overline{y})] \quad (1)$$

where $\circ \in \{+, -, \cdot, /\}$.

Our objective is to estimate the range of a function. Formula (1) provides the exact description of the range of each arithmetic operation over given intervals. During development of various methods that use interval numbers [23–25], it turned out that it is impossible to obtain opposite and inverse element of such numbers. The problem occurs in solving simple interval equation as well as in interval systems of these equations. For this purpose, the extension of classical (proper)