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Numerical solution of a class of delay differential and delay partial differential equations via Haar wavelet

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ABSTRACT

In this paper, Haar wavelet collocation method is applied to obtain the numerical solution of a particular class of delay differential equations. The method is applied to linear and nonlinear delay differential equations as well as systems involving these delay differential equations. In addition to this the method is also extended to numerical solution of delay partial differential equations with delay in time. The method is applied to several benchmark test problems. The numerical results are compared with the exact solutions and the performance of the method is demonstrated by calculating the maximum absolute errors and experimental rates of convergence using different numbers of collocation points. The numerical results show that the method is simply applicable, accurate, efficient and robust.

1. Introduction

Delay differential equations have numerous applications in biological, chemical, electronic and transportation systems [1]. A delay differential equation is a differential equation where the time derivatives at the current time depend on the solution and possibly its derivatives at previous times [2]. Delay differential equations are sometimes also called time delay systems. Specific applications of time-delay systems can be found in chemical processes [3], mechanical systems [4], transmission lines and industrial processes [5]. Several other phenomena like population growth [6], economic growth [7], neural networks [8] etc. can also be modeled using delay-differential equations.

The analysis of time-delay systems is not as straightforward as that of lag-free systems. There exist analytical and numerical methods to analyze delay-systems. Particular concentration has been given to applications of Walsh functions [9], block pulse functions [10], Laguerre polynomials [11], Legendre polynomials [12] and Chebyshev polynomials [13] in the recent past. In 1975, Tsoi obtained the transition matrices in the form of infinite series for a class of delay differential equations with constant coefficients and constant delays [14]. Nagyand and Tikriti suggested a time partition method of analyzing the dynamic responses of linear time-invariant control systems with constant delays [15]. However, this technique couldn't be utilized frankly in the time domain. To take away this limitation, in 1977, Chen and Shih further modified the time-partition method to the time domain to solve linear time-varying delay differential equations with multiple constant time delays. Since the aforementioned techniques are not computer oriented, these are thus not practically valuable.

Recently, applications of orthogonal functions, such as Walsh functions and block pulse functions, to the analysis of delay systems have been reported. In 1978, Shih et al. were the first to utilize Walsh functions to solve delay differential equations [16]. They constructed a Walsh shift matrix to perform right-shift operations. The disadvantages of their method are that

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the number of expansion terms of a Walsh series can't be picked arbitrarily, and it is applicable only to the case of zero initial conditions. To overcome these challenges, in 1982 Chen proposed a generalized Walsh delay operational matrix to solve multiple delays systems using Walsh functions [17]. On the other hand, the use of block-pulse functions considerably simplifies the computational efforts [18,19]. The readers may refer to [20,21] for further development of the theory of delay differential equations.

The use of wavelets has come to prominence during the last two decades. They have wide-ranging applications in scientific computing, and it is no surprise that they have been extensively used in numerical approximation in the recent relevant literature. Some of the recent work using wavelets can be found in the references [22–27].

In the present work, we will use Haar wavelet collocation method. Haar wavelet is based on the functions which were introduced by Hungarian mathematician Alfred Haar in 1910. The Haar wavelet functions are made up of piecewise constant functions and are mathematically the simplest among all the wavelet families. A good feature of this wavelet is the possibility to integrate it analytically arbitrary times. The Haar wavelet has been applied for solving several problems of mathematical calculus. Haar wavelet method is a computer-oriented method, it gives us the possibility to implement standard subprograms. The Haar matrices contain many zeros, this makes the Haar transform faster to compute as compared to other wavelet functions [28].

Specific advantages of the Haar wavelet collocation method are as follows [24]:

- Haar wavelet collocation method uses simple box functions. Consequently the formulation of numerical method based on Haar wavelet is straightforward and involves lesser manual labor.
- Due to better approximating properties of Haar wavelet collocation method, it provides more accurate solutions as compared to other well-established methods.
- Haar wavelet collocation method retains accuracy for the solution as well as its derivatives.
- This method is very convenient for solving boundary value problems, since the boundary conditions are taken into account automatically.
- The Haar wavelet collocation method is particularly suitable for systems involving abruptly varying functions. The Haar wavelet technique is highly feasible for damage detection.

Due to these excellent features of Haar wavelet, the Haar wavelet collocation method is becoming more popular for numerical approximations among researchers.

There are various types of delay differential equations. In the present work, we will consider the following equations and systems:

- Time-invariant and time-varying delay differential equations
- Delay differential system
- Delay partial differential equations

A time-invariant delay system is one in which the system output obtained from the system input does not vary with time, that is, it is a system whose output does not depend explicitly on time. For many mechanical and electrical systems it is important to know whether the system is time-invariant or not [29]. Time-invariant delay systems, often known as delay systems, can be characterized by the ordinary nth-order differential equations with constant coefficients. Time invariant delay systems represent a class of infinite-dimensional systems which appear naturally in a large number of systems such as communication networks, remote controls, mass transports, chemical processes and biological systems in population dynamics [29].

A time-varying system is a system that is not time invariant. The equations characterizing time-varying systems are similar to those characterizing time-invariant systems with the exception that the coefficients can be functions of time [30]. Applications of time-varying delay systems include the microphones containing a variable capacitor and the induction generators in which the mutual inductance between the primary and secondary windings is variable [30].

In the present work, we will consider delay differential equations with time-delay τ in the state and control of the following form [31]:

$$\begin{cases} \dot{u}(t) = a(t)u(t) + b(t)v(t) + c(t)u(t-\tau) + d(t)v(t-\tau), \\ u(0) = u_0, \\ u(t) = \phi(t), \quad -\tau \le t < 0, \end{cases} \tag{1}$$

where $u(t)$ is a state function, $v(t)$ is a control function, $u(0)$ is the initial condition and $\phi(t)$ is the delay condition. The coefficients $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are functions of time in the case of time-varying delay differential equations. On the other hand, if the coefficients $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are constants, then we get a time-invariant delay differential equation.

We will also consider systems of delay differential equations with a time-delay τ in state in the following form [18,29]:

$$\begin{cases} \dot{\mathbf{u}}(t) = \mathbf{A}(t)\mathbf{u}(t) + \mathbf{B}(t)\mathbf{v}(t) + \mathbf{F}(t)\mathbf{u}(t-\tau) \\ \mathbf{u}(0) = \mathbf{u}_0 \\ \mathbf{v}(t) = \boldsymbol{\phi}(t), \qquad -\tau \le t < 0, \end{cases} \tag{2}$$

where $\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T$ is an n-dimensional state vector, $A(t)$, $B(t)$ and $F(t)$ are matrices of time functions of suitable dimension, $\mathbf{v}(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^T$ is an n-dimensional control vector, $\mathbf{u}_0$ is the initial condition and $\boldsymbol{\phi}(t)$ is the delay condition.