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# Fourier series solutions for vibrations of a rectangular plate with a straight through crack

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ABSTRACT

The existence of a crack in a rectangular plate makes the exact closed-form solutions for the vibrations of the plate intractable, if they even exist. This work presents analytical solutions for vibrations of horizontally or vertically cracked rectangular plates under various boundary conditions. The solutions are constructed by combining Fourier cosine series with domain decomposition. A rectangular plate with a side crack or an internal crack is divided into four or six rectangular sub-domains, respectively. The series solutions that satisfy the governing equations for the vibrations of a plate are firstly established for each sub-domain based on the classical plate theory. The solutions for each sub-domain are related to each other by satisfying the continuity conditions along the interconnection boundaries between the sub-domains. Finally, the boundary conditions of the cracked plate are imposed on the solutions. Comprehensive convergence studies are performed for intact plates and cracked plates with various boundary conditions, and the presented natural frequencies are compared with the published ones to confirm the correctness of the proposed solutions. Not like a typical energy method, which overpredicts true frequencies if the used shape functions do not exactly satisfy the governing equations for the problem under consideration, the convergence studies reveal that the present solutions converge from the lower bounds to the exact frequencies as the number of series terms increases. The present solutions are further applied to determine the first five frequencies of vibration of rectangular plates with side cracks and internal cracks of various lengths and locations. The results obtained under SSSS, CFFF, FSFS and FFFF boundary conditions are tabulated, and some of these are presented here for the first time.



## 1. Introduction

Plates are extensively used as structural elements in engineering designs. Their vibrational behavior – especially their free vibration characteristics (natural frequencies and their corresponding mode shapes) – is of great interest, and a great number of research papers have been published on this topic. A crack in a plate causes a local change in its stiffness and may significantly alter the dynamic characteristics of the plate. Accordingly, investigation of the vibrations of a cracked plate is interesting and important. Numerous studies of vibrations of cracked plates, based on the classical plate theory, have been published.

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A crack makes exact closed-form solutions for the vibrations of a rectangular plate intractable, even under simply supported boundary conditions. Numerical techniques such as finite element approaches and the Ritz method are popular for solving such problems. Theoretically, $C^1$-type elements, which are difficult to establish, should be used in a finite element solution when the classical plate theory is applied. Qian et al. [1] and Krawczuk [2] developed their finite element solutions by establishing a stiffness matrix of an element that includes the crack tip, by integrating the stress intensity factor. Ma and Huang [3] simply employed the commercial finite element computer program ABAQUS with eight-node two-dimensional shell elements (S8R5) to determine natural frequencies of cracked rectangular plates.

Several studies have employed the Ritz method to vibrations of cracked rectangular plates. The Ritz method is an effective method that is commonly applied to approximate the modal characteristics of structures. However, the selection of appropriate admissible functions in the method is essential for obtaining accurate solutions before numerical difficulties arise. Decomposing a cracked rectangular plate into several sub-domains, whose displacement functions are approximated using orthogonal polynomial functions, Yuan and Dickinson [4] introduced artificial springs at the interconnecting boundaries between the sub-domains, while Liew et al. [5] required continuities of displacement and slope in a sense of integration along the interconnecting boundaries. These decomposition techniques fail to provide strict continuities of displacement and slope along the interconnecting boundaries and destroy the favorable characteristics of the Ritz method in providing upper-bound solutions for vibration frequencies. Khadem and Rezaee [6] proposed solutions for vibrations of a simply-supported rectangular plate with a horizontal crack using so-called modified comparison functions that were constructed from Levy's solution as the admissible functions in the Ritz method. To describe appropriately the stress singularities at a crack tip and the discontinuities of displacement and slope across a crack, Huang and his co-workers [7,8] incorporated crack functions into the admissible functions, which also consist of orthogonal polynomial functions.

Integration equation approaches have been employed to solve the differential equations that govern the vibrations of cracked rectangular thin plates under simply supported conditions at all edges or two opposite edges and having cracks parallel to one of the edges. Lynn and Kumbasar [9] applied a Green's function approach to obtain Fredholm integral equations of the first kind. Stahl and Keer [10] and Aggarwala and Ariel [11] used double series to establish homogeneous Fredholm integral equations of the second kind. Neku [12] employed Levy's form of solution to establish the needed Green's functions in the approach of Lynn and Kumbasar [9]. Solecki [13] applied double Fourier sine series and the generalized Green-Gauss theorem to form a system of integration equations.

Although many numerical solutions have been developed for the vibrations of cracked plates, analytical solutions are still sought. The main purpose of the work is to present analytical solutions for the vibrations of rectangular plates under various boundary conditions and with cracks that are parallel to one of the edges based on the classical plate theory. The analytical solutions are established by decomposing the cracked rectangular plate of interest into various sub-domains and using Fourier cosine series with eight auxiliary polynomial functions to satisfy the governing equation and boundary conditions in each sub-domain and the continuity conditions along the interconnecting boundaries between adjacent sub-domains. Theoretically, the governing equations, boundary conditions and continuity conditions are exactly satisfied when the number of terms in the solution is sufficiently large. Several numerical examples are solved herein to demonstrate the excellent accuracy and convergence of the present solution. The correctness of the present solution is verified by comparing it with the exact solutions for an intact rectangular plate under simply supported boundary conditions and the published numerical solutions for cracked rectangular plates with various boundary conditions. The present solutions are further applied to determine the first five frequencies of vibration of rectangular plates with various boundary conditions and with side cracks and internal cracks of various lengths and locations.

## 2. Series solutions to governing equations

Consider a rectangular plate with a side crack or an internal crack as shown in Fig. 1. Side-cracked and internally cracked plates are decomposed into four and six rectangular sub-domains, respectively. For each sub-domain, the differential equation that governs the vibrations of the plate is,

$$D\left(\frac{\partial^4 w^{(j)}(x_j,y_j)}{\partial x_j^4}+2\frac{\partial^4 w^{(j)}(x_j,y_j)}{\partial x_j^2\partial y_j^2}+\frac{\partial^4 w^{(i)}(x_j,y_j)}{\partial y_j^4}\right)-\rho h\omega^2 w^{(j)}(x_j,y_j)=0, \tag{1}$$

where $w^{(j)}$ and $(x_j, y_j)$ are the flexural displacement and the local coordinates in sub-domain $j$, respectively; $\omega$ is the circular frequency; $D$, $\rho$ and $h$ are the flexural rigidity, density and thickness of the plate, respectively.

Li et al. [14] demonstrated that the following series can be used to solve Eq. (1).

$$w^{(j)}\left(x_j,y_j\right)=\sum_{m=0}^{M_j}\sum_{n=0}^{N_j}A_{mn}^{(j)}\cos\lambda_m^{(j)}x_j\cos\lambda_n^{(j)}y_j+\sum_{i=1}^{4}\left(\sum_{n=0}^{N_j}C_{in}^{(j)}P_{ix_j}^{(j)}\left(x_j\right)\cos\lambda_n^{(j)}y_j+\sum_{m=0}^{M_j}B_{im}^{(j)}P_{iy_j}^{(j)}\left(y_j\right)\cos\lambda_m^{(j)}x_j\right), \tag{2}$$

where $j$ = 1,2, ..., $J$, and $J$ = 4 or 6; $\lambda_m^{(j)}=m\pi/a_j$; $\lambda_n^{(j)}=n\pi/b_j$; $a_j$ and $b_j$ are the lengths of the sides of the rectangular sub-domain $j$; $A_{mn}^{(j)}$, $B_{im}^{(j)}$ and $C_{in}^{(j)}$ are coefficients to be determined; the supplementary functions $P_{ix_j}^{(j)}(x_j)$ and $P_{iy_j}^{(j)}(y_j)$ satisfy