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A new linear Naghdi type shell model for shells with little regularity

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ABSTRACT

In this paper, a new linear shell model of Naghdi type is formulated for shells with little regularity, namely the shells whose middle surface is parameterized by a $W^{1,\infty}$ function. Thus corners in the undeformed geometry are inherent in the formulation.

Unknowns in the model are the displacement $\tilde{\boldsymbol{u}}$ of the middle surface of the shell and the infinitesimal rotation $\tilde{\boldsymbol{\omega}}$ of the shell cross-section. In difference to the classical shell models, the existence and uniqueness of the solution is obtained for $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\omega}}) \in H^1 \times H^1$ with a very simple proof without usage of almost any differential geometry of surfaces.

We relate the new model with known shell models in two ways. In the first we show that asymptotically, with respect to small thickness of the shell, the model behaves as the membrane model or the flexural shell model in the corresponding regime. In the second, for smooth enough middle surface, we relate the terms in the weak formulation of the model with terms in the classical Naghdi shell model. Further, we prove continuous dependence of the solution of the model on the change of the undeformed middle surface. At the end, we also present a numerical approximation of the model for the middle surface with a corner at the joint of two pieces that dominantly behave differently (one as a membrane and the other one as a flexural shell).

1. Introduction

In this work we propose a new linear shell model of Naghdi type and give the arguments for well soundedness of its formulation. The model has been announced in [1] where a model of Koiter type has been analyzed. However, the function space for the Naghdi type model is simpler and simplifies the numerical treatment and some of the analysis. There are many shell models in use (membrane, flexural, Koiter, Budansky–Sanders, Naghdi etc.). They are all given by two-dimensional equations that describe behavior of the middle surface of the thin three-dimensional body. Their formulations differ in several features, e.g. unknown variables used in their definition (displacement, rotation), behavior they capture (membrane, bending, shear), the global or local unknowns (unknowns are components in local basis, tangential and normal displacements (v_1, v_2, v_3) or physical vector displacement $\tilde{\boldsymbol{v}}$), smoothness of the middle surface of the shell necessary for the model (C^3, $W^{2,\infty}$ or $W^{1,\infty}$), function space of the formulation (a subset of H^2 or H^1 for the normal displacement), etc.

Especially important is the way the model has been related to the three-dimensional elasticity for the associated three-dimensional body, see [2]. In a series of papers Ciarlet, Lods and Miara derived and justified, in a mathematically rigorous

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way, the following linear models: the elliptic membrane model in [3], the flexural model in [4], the generalized membrane model in [5] and the Koiter model in [6]. Justification for the first three models is done directly from the three-dimensional elasticity. Namely they showed that, in a suitable regime, the limit of solutions of the associated three-dimensional linearized elasticity problem for the thin shell-like body when the thickness h of the body tends to zero satisfies the corresponding model. The membrane effects are of order h, while flexural are of order h^3 and the above limiting procedures produce only models with single order effects. However, the Koiter and Naghdi models contain effects of different orders of h. Thus, a different strategy is applied in [6] to justify the Koiter model. It was shown that the Koiter model in each regime asymptotically behaves as the membrane, the flexural and the generalized membrane shell. In the same spirit it was shown in [7] that the underlying shell model behind the general shell elements compares well with the classical shell models since it displays the same asymptotic behaviors. This approach we take and relate the proposed model to the membrane and flexural shell model. The same is done in [1] for the model of the Koiter type.

As mentioned above, another important issue is necessary smoothness for the model to be well defined. The classical assumption is that the middle surface of the shell is parametrized by a C^3 function since the derivation of the shell models from three-dimensional equations is done in local coordinates attached to the shell. In order to relax smoothness the idea is to rewrite models in terms of physical vector displacements (rotations) and not in the local basis related to the surface. This is done for the classical Koiter shell model in [8] and for the Naghdi shell model in [9] for $W^{2,\infty}$ middle surfaces. In [10] the vector of infinitesimal rotation is used in the formulation of the model (see also [11]) and the existence and uniqueness of the solution was proved for $W^{1,\infty}$ shells whose normal vector is in $W^{1,\infty}$ too, see also [12]. A different definition of the infinitesimal rotation vector from [13] allows formulation of the curved rod model for $W^{1,\infty}$ rods. Inspired by this different definition of the infinitesimal rotation vector $\tilde{\boldsymbol{\omega}}$, which also accounts for the rotation of the cross–section around its axes (drilling degree of freedom), the flexural shell model is rewritten in [14] and the model of the Koiter type is formulated in [1] for shells with $W^{1,\infty}$ middle surface. These formulations serve us as a motivation for the definition of a shell model of the Naghdi type. Note also that inclusion of drilling rotations in shell models is not new in the literature, see [15–17].

The Naghdi shell model is a two-dimensional shell model, see [18]. The unknowns of the problem are the displacement of the points of the shell midsurface $\tilde{\boldsymbol{u}}$ and the rotation field of the normal to the midsurface $\tilde{\boldsymbol{\omega}}$ (classically not including twist). The energy of Naghdi's shell model is consisting of three parts: the membrane energy, the flexural energy and the transverse shear energy. The first two terms are the same as in the energy of the Koiter shell model. Namely, in contrast to the Koiter shell model, cross–sections are allowed to change the angle with respect to the deformed middle surface, but this change is "penalized" in the transverse shear energy term. In this spirit we propose a shell model of the Naghdi type valid for $W^{1,\infty}$ middle surfaces: find $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\omega}}) \in V_N(\omega)$ such that:

$$h\int_\omega \mathbf{Q}\mathcal{C}_m\mathbf{Q}^T\left[\ \partial_1\tilde{\boldsymbol{u}}+\boldsymbol{a}_1\times\tilde{\boldsymbol{\omega}}\quad \partial_2\tilde{\boldsymbol{u}}+\boldsymbol{a}_2\times\tilde{\boldsymbol{\omega}}\ \right]\cdot\left[\ \partial_1\tilde{\boldsymbol{v}}+\boldsymbol{a}_1\times\tilde{\boldsymbol{w}}\quad \partial_2\tilde{\boldsymbol{v}}+\boldsymbol{a}_2\times\tilde{\boldsymbol{w}}\ \right]\sqrt{a}dx$$
$$+\frac{h^3}{12}\int_\omega \mathbf{Q}\mathcal{C}_f\mathbf{Q}^T\nabla\tilde{\boldsymbol{\omega}}\cdot\nabla\tilde{\boldsymbol{w}}\sqrt{a}dx=\int_\omega \tilde{\boldsymbol{f}}\cdot\tilde{\boldsymbol{v}}\sqrt{a}dx,\quad (\tilde{\boldsymbol{v}},\tilde{\boldsymbol{w}})\in V_N(\omega). \tag{1.1}$$

where, h is the thickness of the shell, $\boldsymbol{a}_1, \boldsymbol{a}_2$ span tangential plane of the shell, $\mathbf{Q}\mathcal{C}_m\mathbf{Q}^T$ and $\mathbf{Q}\mathcal{C}_f\mathbf{Q}^T$ are tensors depending on the geometry and the material of the shell, see (2.3).

The main features of the model are:

(i) the model is formulated for the unknown $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\omega}})$ in a subset $V_N(\omega)$ of $H^1(\omega;\mathbb{R}^3)\times H^1(\omega;\mathbb{R}^3)$ defined by boundary conditions ($\omega\subset\mathbb{R}^2$ being open, bounded with Lipschitz boundary),
(ii) the proof of existence and uniqueness of the solution (Theorem 2.2) is based on an estimate (Lemma 2.1) with a very simple proof without use of delicate Korn's type estimate,
(iii) the model is well defined for the middle surface parameterized by $\boldsymbol{\varphi}\in W^{1,\infty}(\omega;\mathbb{R}^3)$ (and thus the model for shells with middle surfaces with corners (or folded plates or shells) is inherently built into the new model),
(iv) the energy of the model contains the membrane, transverse shear and flexural terms which are of different order with respect to the thickness h of the shell,
(v) for smooth geometry the solution of the model in the elliptic membrane and flexural regime tends to the solution of the corresponding shell model, when thickness h tends to zero (see Section 3),
(vi) the model can be seen as a small perturbation of the classical Naghdi shell model (see Section 4),
(vii) the solution of the model continuously depends on the change in the geometry (with respect to parametrization $\boldsymbol{\varphi}$ in $W^{1,\infty}(\omega;\mathbb{R}^3)$) (see Section 5),
(viii) the model can be seen as the special Cosserat shell model with a single director (see [19]) for a particular linear constitutive law (see Section 6).

Note that the Koiter type model from [1] is posed by the same equation, i.e. (1.1), but on a nontrivial subspace of $V_N(\omega)$, see Section 2.2 for more details. This results in a simpler mathematical structure of the model (1.1). Moreover, for the model of the Koiter type from [1] properties iv), vi) and vii) have not been proved. The simple form of the model (1.1) eases the analysis and derivation of it properties.

The solution of one problem for a shell with a corner and consisting of two parts, one which is behaving dominantly as a membrane shell and one behaving dominantly as a flexural shell is demonstrated in Fig. 1. In general any 2D shell model gives only an approximation of the solution of 3D equations. In the case of the midsurface with corners there are some