ARTICLE IN PRESS

Applied Mathematical Modelling xxx (2014) xxx–xxx

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

The construction of operational matrix of fractional integration using triangular functions

K. Maleknejad [a,*], M. Asgari [b]

[a] Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
[b] Department of Engineering, Abhar Branch, Islamic Azad University, Abhar, Iran

ARTICLE INFO

Article history:
Received 21 November 2012
Received in revised form 11 April 2014
Accepted 1 September 2014
Available online xxxx

Keywords:
Abel's integral equations
Fractional calculus
Integro-differential equation
Multi-order fractional differential
Triangular functions

ABSTRACT

In this paper the operational matrix of triangular functions for fractional order integration in the Caputo sense is derived. Also, the operational matrix of fractional integration is applied for solving multi-order fractional differential equations, Abel's integral equations and nonlinear integro-differential equations. This technique is a successful method because of reducing such problems to solve a system of algebraic equations; so, the problem can be solved directly. The advantage of this method is low cost of setting up the equations. Illustrative examples demonstrate accuracy and efficiency of the method.

1. Introduction

In recent years a lot of attention has been devoted to study the fractional calculus. The reason is that physical phenomena which dependence on the time instant and on the previous time history can be provided more accurate models by using fractional calculus [1–4]. Most of these equations cannot be solved analytically; therefore, we have to use numerical techniques. Some of these numerical methods are He's variational iteration method [5], homotopy perturbation method [6], fractional transform method [7], collocation method [8], Taylor expansion method [9], Chebyshev wavelet method [10] and Cas wavelet method [11].

For differential and integral equations of integer order, the orthogonal functions have been applied for solving various problems numerically. Some examples are block pulse functions (BPFs) [12,13], triangular functions (TFs) [14–16], Legender polynomials [17,18] and Haar wavelets [19]. By using truncated orthogonal series and their operational matrices the problem reduces to a system of algebraic equations; thus, we can solve the problem by iteration methods.

In the present paper we construct operational matrix of triangular functions for fractional integration to solve fractional multi-order differential equations, nonlinear integro-differential equations of fractional order and Abel's integral equations. TFs are proposed by Deb et al. [20,21]. They established that TF technique is more accurate than block pulse technique. The advantage of TF is that TF representation dose not need any integration to evaluate the coefficients, thus reducing a lot of computational cost.

The paper is ordered as follows:

* Corresponding author.
E-mail addresses: maleknejad@iust.ac.ir (K. Maleknejad), mah_sgr@yahoo.com (M. Asgari).

http://dx.doi.org/10.1016/j.apm.2014.09.009
0307-904X/

In Section 2, a brief review of the TFs and fractional calculus is presented. In Section 3, operational matrices of TFs for fractional integration are derived. Sections 4 and 5 are devoted to the formulation of multi-order fractional differential equations, integro-differential equations of fractional order and Abel's integral equations. In Section 6, convergence analysis is discussed. Some numerical examples are provided in Section 7. Finally, Section 8 gives a brief conclusion.

2. Basic definitions

2.1. Fractional calculus

In this section, we give some definitions and properties of fractional calculus theory.

Definition 2.1. Let $f \in L_1, \alpha \in R_+$. The Riemman–Liouville fractional integral of f of order α is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \ t > 0.$$

Definition 2.2. The Caputo fractional derivative is given by

$$D^{\alpha}f(t) = \begin{cases} I^{n-\alpha}f^{(n)}(t) & n-1 < \alpha < n, \\ \frac{d^n}{dt^n}f(t). & \alpha = n. \end{cases}$$

Some properties of the fractional operator are mentioned from [22]

(1) $D^{\alpha}x^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, & \beta \geqslant \lceil \alpha \rceil, \\ 0, & \beta < \lceil \alpha \rceil. \end{cases}$

(2) $D^{\alpha}I^{\alpha}f(t) = f(t)$,

(3) $I^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0^+)\frac{t^i}{i!}, \ t > 0, \ , n-1 \leqslant \alpha < n$,

(4) $I^{\alpha}I^{\beta}f(t) = I^{\alpha+\beta}$,

(5) $I^{\alpha}I^{\beta}f(t) = I^{\beta}I^{\alpha}$.

2.2. TFs and their properties

Two m-set TFs presented by Deb et al. [21] are defined over the interval [0,T) as follows

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h} & ih \leqslant t < (i+1)h, \\ 0 & elsewhere, \end{cases}$$

$$T2_i(t) = \begin{cases} \frac{t-ih}{h} & ih \leqslant t < (i+1)h, \\ 0 & elsewhere, \end{cases}$$

where, $i = 0, \ldots, m-1$, and $h = \frac{T}{m}$.

For the whole set of BPFs, $\phi_i(t)$, we have

$$\phi_i(t) = T1_i(t) + T2_i(t),$$

where,

$$\phi_i(t) = \begin{cases} 1 & ih \leqslant t < (i+1)h, \\ 0 & elsewhere, \end{cases}$$

TFs, are disjoint, orthogonal, and complete [14].

Now, we define m-set TF vectors as

$$T1(t) = [T1_0(t), \ldots, T1_{m-1}(t)]^T, \quad T2(t) = [T2_0(t), \ldots, T2_{m-1}(t)]^T,$$

and

$$T(t) = [T1(t), T2(t)]^T.$$

In general, a square integrable function $f(t)$ may be expanded into an m-set TF series as:

$$f(t) \simeq \hat{f}(t) = F1^T T1(t) + F2^T T2(t) = F^T T(t), \quad t \in [0, T), \tag{1}$$

where, $F1_i = f(ih)$ and $F2_i = f((i+1)h)$ for $i = 0, \ldots, m-1$. The vectors $F1$ and $F2$ are called the 1D-TF coefficient vectors and $2m$-vector F is defined as:

Please cite this article in press as: K. Maleknejad, M. Asgari, The construction of operational matrix of fractional integration using triangular functions, Appl. Math. Modell. (2014), http://dx.doi.org/10.1016/j.apm.2014.09.009