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A reliability model for a three-state degraded system having random degradation rates

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ABSTRACT

For degraded multi-state systems, it has been assumed in the literature that, for any given system, the instantaneous degradation rates are fixed. This paper attempts to study a three-state degraded system that have random degradation rates among its states. In particular, a reliability model for such a three-state system is presented assuming that the degradation rates are random and statistically dependent. The dependence is modeled by copulas, and dynamic reliability analysis of the system is performed. Graphical illustrations are provided, and comparisons are made with the corresponding results for the classical fixed rates model.

1. Introduction

Multi-state systems can perform their tasks with various distinguished levels of efficiency, usually referred to as performance rates [1]. One of the most suitable approaches for modeling a multi-state system in a dynamic way is based on Markov processes. In Liu and Kapur [2], time dependent system state probabilities have been obtained assuming that the degradation in the multi-state system follows a Markov process. In their derivations, degradation rates are denoted by the parameters $\lambda_{r,s}$, where $\lambda_{r,s}$ is the instantaneous degradation rate from state r to a lower state s, for $r > s$. Thus, the system state probabilities are represented as a function of the parameters $\lambda_{r,s}$. An extensive discussion on stochastic process based modeling of multi-state system is presented in [3]. Various methods have been applied for dynamic reliability evaluation of multi-state systems. The combined universal generating function and stochastic process method have been developed for dynamic analysis of multi-state systems [3]. Xue and Yang [4] studied some dynamic reliability characteristics for multi-state systems by combining Markov processes with multi-state reliability theory. Eryilmaz [5] defined and studied mean residual and mean past lifetime functions for multi-state systems. Sheu and Zhang [6] studied multi-state systems when the degradation in multi-state elements follows a non-homogenous Markov process. Their method is based on the *Lz*-transform method. For the details of the *Lz* -transform method, we refer to Lisnianski [7]. Other recent studies on dynamic analysis of multi-state systems are [8–10].

Generally, it has been assumed in the literature that, for any given system, the instantaneous degradation rates $\lambda_{r,s}$ are constant. However, within certain populations, the degradation rates λ_{rs} can vary from system to system even if the systems have been setup or produced by the same supplier. For example, a complex system is operated in a random environment and the magnitudes of λ_{rs} change as the environment changes. Such situations are also common for household items. For example, a washing machine of the same brand that is used by different families might have different rates of degradation. Thus, modeling the rates λ_{rs} as random variables may lead to make a more comprehensive time dependent reliability analysis that considers deviations in λ_{rs} for the systems/units used under different conditions.

In this paper, we present a reliability model for a three-state degraded system when the degradation in system follows a homogeneous Markov process having random degradation rates (see, e.g. [11] for three-state systems). Such a model is described by two degradation rates λ_{10} and λ_{21} which are assumed to be random and statistically dependent. Although the paper deals with the special case when the system has three states, as stated in Section 5, the model can be extended to the systems having multi states. Our method for modeling dependence between λ_{10} and λ_{21} is based on copulas. Copulas provide a way of specifying joint distribution of random variables if only the marginal distributions of the random variables are known. A detailed review of copulas can be found in [12].

The present paper is organized as follows. In Section 2, we present main definitions, assumption, and preliminary results that will be helpful throughout the paper. Section 3 involves modeling with Farlie–Gumbel–Morgenstern copula. In Section 4, we obtain some ordering relations for comparing two independent systems based on their degradation rates. Section 5 contains a numerical example to illustrate the findings of the paper.

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Nomenclature

0	complete failure state
1	partially working state
2	perfect functioning state
λ_{10}	instantaneous degradation rate from state "1" to state "0"
λ_{21}	instantaneous degradation rate from state "2" to state "1"
$H(x_1, x_2)$	the joint cumulative distribution function of λ_{10} and λ_{21}
$h(x_1, x_2)$	the joint probability density function of λ_{10} and λ_{21}
C	copula
FGM	Farlie–Gumbel–Morgenstern
T_1	time spent by the system in state "1"
T_2	time spent by the system in state "2"

2. Definitions, assumptions and preliminaries

Throughout the paper we assume the following for the system under consideration.

1. The system can be in one of three possible states 0,1,2 at any time, where the extreme states "0" and "2" represent the completely failed and completely working states respectively, and the state "1" represents partially working state.

2. The state set $\{0, 1, 2\}$ of the system are ordered, and the states are disjoint.

3. The system degrades with time t from the perfect state to the lower states, and only minor failures occur in the system. That is, the system cannot transit from the state "2" to the state "1".

4. The system is nonrepairable.

Under the assumption that the degradation in the system follows a homogeneous Markov process, the time T_2 spent by the system in state "2" and the time T_1 that is spent by the system in state "1" are independent, and $P\{T_1 > t\} = e^{-\lambda_{10}t}$ and $P\{T_2 > t\} = e^{-\lambda_{21}t}$, $t \geq 0$ [13]. That is, the sojourn time distribution from state "1" to state "0" is exponential with mean $1/\lambda_{10}$, and from state "2" to "0" it is exponential with mean $1/\lambda_{21}$, and both durations are independent.

It is reasonable to assume a statistical dependence between the random variables λ_{10} and λ_{21} since for example larger values of λ_{21} may lead to a faster transition from partially working state "1" to completely failed state "0". The degree of dependence between λ_{10} and λ_{21} heavily depends on the system and the definition of its states.

Let the joint cumulative distribution function of $(\lambda_{10}, \lambda_{21})$ is given by

$$H(x_1, x_2) = P\{\lambda_{10} \leq x_1, \lambda_{21} \leq x_2\}.$$

Assume that the dependence between λ_{10} and λ_{21} is modeled by a copula function C: $[0, 1]^2 \to [0, 1]$ such that

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2)), \tag{1}$$

where $F_1(x) = P\{\lambda_{10} \leq x\}$ and $F_2(x) = P\{\lambda_{21} \leq x\}$ are the marginal distribution function of λ_{10} and λ_{21}, $0 < x < 1$. The supports of the distributions F_1 and F_2 are chosen to be $(0, 1)$ since it is reasonable to assign a value for the rates from the unit interval. Under these assumptions, the random variables T_1 and T_2 are no longer independent. However, they are conditionally independent given λ_{10} and λ_{21}, and

$$P\{T_1 > t_1, T_2 > t_2 | \lambda_{10} = x_1, \lambda_{21} = x_2\} = e^{-x_1 t_1} e^{-x_2 t_2}.$$

Thus the joint survival function of (T_1, T_2) can be written as

$$P\{T_1 > t_1, T_2 > t_2\} = \int_0^1 \int_0^1 P\{T_1 > t_1, T_2 > t_2 | \lambda_{10} = x_1, \lambda_{21} = x_2\} h(x_1, x_2) dx_2 dx_1 = \int_0^1 \int_0^1 e^{-x_1 t_1} e^{-x_2 t_2} h(x_1, x_2) dx_2 dx_1, \tag{2}$$

where $h(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} H(x_1, x_2)$ is the joint probability density function (pdf) of λ_{10} and λ_{21}. We have the following expression for $h(x_1, x_2)$.

$$h(x_1, x_2) = c(F_1(x_1), F_2(x_2)) f_1(x_1) f_2(x_2), \tag{3}$$

where $c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2)$, and $f_i(x)$ is the pdf associated with $F_i(x)$, $i = 1, 2$. Thus substituting (3) in (2), the joint survival function of (T_1, T_2) can be rewritten as

$$P\{T_1 > t_1, T_2 > t_2\} = \int_0^1 \int_0^1 e^{-x_1 t_1} e^{-x_2 t_2} c(F_1(x_1), F_2(x_2)) f_1(x_1) \times f_2(x_2) dx_2 dx_1 \tag{4}$$

In Section 3, we derive explicit expressions for the joint and marginal survival functions of T_1 and T_2 when the copula function C belongs Farlie–Gumbel–Morgenstern (FGM) family of copulas and the random variables λ_{10} and λ_{21} are modeled by power distributions.

3. Modeling with FGM copula

There are many copulas that can be used for modeling the dependence between the random variables. A fundamental issue of using copulas is how to choose an appropriate copula to model the dependence structure between the random variables. Each copula has its own dependence structure and properties. These properties should be taken into account in the model construction.

Let the copula function in (1) is given by

$$C(u_1, u_2) = u_1 u_2 \{1 + \alpha(1 - u_1)(1 - u_2)\}, \tag{5}$$

for $|\alpha| \leq 1$, and $0 \leq u_i \leq 1$, $i = 1, 2$. The copula given by (5) is known to be FGM copula, it allows for modeling both positive dependence (when $\alpha \geq 0$) and negative dependence (when $\alpha \leq 0$) [14]. FGM type copula has been used in the context of reliability due to its mathematically tractable form [15,16].

Assume that the random variables λ_{10} and λ_{21} have power distributions with cdfs

$$F_1(x) = P\{\lambda_{10} \leq x\} = x^{a_1}, \quad 0 < x < 1, a_1 > 1,$$

and

$$F_2(x) = P\{\lambda_{21} \leq x\} = x^{a_2}, \quad 0 < x < 1, a_2 > 1.$$

Then from (3), the joint pdf of λ_{10} and λ_{21} is obtained as

$$h(x_1, x_2) = a_1 a_2 x_1^{a_1 - 1} x_2^{a_2 - 1} \left[1 + \alpha(1 - 2x_1^{a_1})(1 - 2x_2^{a_2})\right]. \tag{6}$$

Thus using (4), the joint survival function of (T_1, T_2) can be computed as follows.