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Dynamic analysis of a fractional order prey–predator interaction with harvesting

M. Javidi *, N. Nyamoradi

Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran

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ABSTRACT

In recent years, prey–predator models appearing in various fields of mathematical biology have been proposed and studied extensively due to their universal existence and importance. In this paper, we introduce a fractional-order prey–predator model and deals with the mathematical behaviors of the model. The dynamical behavior of the system is investigated from the point of view of local stability. We also carry out a detailed analysis on the stability of equilibrium. Numerical simulations are presented to illustrate the results.

1. Introduction

In recent years, population models appearing in various fields of mathematical biology have been proposed and studied extensively due to their universal existence and importance [1]. Among the most widely used population models in theoretical ecology, the Holling–Tanner model plays a special role in view of the interesting dynamics it possesses. This model has been widely studied by several researchers, for instance [2–7].

The most crucial element in prey–predator models is the "functional response" or "trophic function", the function that describes the number of prey consumed per predator per unit time for given quantities of prey and predator. Various forms of functional responses have become the focus of considerable attention from time to time in ecological literature. The most important and useful functional response is the so-called Michaelis–Menten or Holling type-II functional response of the form $V(x) = \frac{cx}{m+x}$, where x and y are the population densities of the prey and predator, respectively; c is the maximal predator per capita consumption rate, i.e. the maximum number of prey that can be eaten by a predator in each time unit and m is the half capturing saturation constant, i.e. the number of prey necessary to achieve one-half of the maximum rate c. Many species models with such functional responses are extensively studied in ecological literature [8–10].

In [11], a modified Holling–Tanner prey–predator model with time delay is considered. By regarding the delay as the bifurcation parameter, the local asymptotic stability of the positive equilibrium is investigated. The authors of [12], studied a ratio-dependent prey–predator model with the Allee effect in the growth of the prey population. They analyse the stability properties of the system, present a complete bifurcation analysis and show all possible non-degenerated phase portraits. In [13], a delayed stage-structured prey–predator model with non-monotone functional responses is proposed. It is assumed that immature individuals and mature individuals of the predator are divided by a fixed age, and that immature predators do not have the ability to attack prey. In [14], the main feature is that the authors introduce time delay and pulse into the

* Corresponding author. Tel./fax: +98 0831 4274569.
E-mail address: mo_javidi@yahoo.com (M. Javidi).

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prey–predator (natural enemy–pest) model with age structure, exhibit a new modeling method which is applied to investigate impulsive delay differential equations, and give some reasonable suggestions for pest management. The authors of [15], have studied changes in the dynamics of a predator population, which otherwise lives on a native prey, in presence of migratory prey that carries some infection. They study predicts one of the four behaviors when some system parameters were varied: eradication of the disease, predator extinction, coexistence at stable equilibrium, or coexistence on limit cycles.

Lotka [16] in 1925 and Volterra [17] in 1926 introduced the first predator-prey model. After that many more complicated but realistic PP models have been formulated by ecologists and mathematicians. One of the most popular prey–predator models was introduced by Freedman in 1980 (see [18] for more details), which has the Michaelis–Menten type functional response. The authors of [19], studied a simple prey–predator interaction where predator population is subject to harvesting as follows

$$\begin{cases} \frac{dP(t)}{dt} = sP(t)\left(1 - \frac{P(t)}{K}\right) - \alpha\frac{P(t)N(t)}{1+\alpha_1 P(t)}, \\ \frac{dN(t)}{dt} = \beta\frac{P(t)N(t)}{1+\alpha_1 P(t)} - s_0 N(t) - EN(t), \end{cases} \tag{1}$$

with the following initial conditions: $P(0) > 0, N(0) > 0$. Let $P(t)$ and $N(t)$ be the prey and predator densities at time t respectively. Assume that the prey population grows logistically to its carrying capacity K with intrinsic growth rate s in absence of predator. Let d_0 be the food-independent death rate and α, α_1, s_0, E are positive real numbers.

In this paper, we investigate a fractional order prey–predator interaction with harvesting as the following form:

$$\begin{cases} \frac{d^\theta P(t)}{dt^\theta} = sP(t)\left(1 - \frac{P(t)}{K}\right) - \alpha\frac{P(t)N(t)}{1+\alpha_1 P(t)}, \\ \frac{d^\theta N(t)}{dt^\theta} = \beta\frac{P(t)N(t)}{1+\alpha_1 P(t)} - s_0 N(t) - EN(t) \end{cases} \tag{2}$$

and with the following initial conditions: $P(\delta) > 0, N(\delta) > 0$, where $0 < \theta \leqslant 1$ is a real number, $\frac{d^\theta}{dt^\theta}$ is the standard Caputo(C) differentiation.

Fractional calculus is the area of mathematics that extends derivatives and integrals to an arbitrary order (real or, even, complex order) which emerged at the same time as the classical differential calculus [20–26]. Bagley and Torvik [27–29] provided a review of work done in this area prior to 1980, and showed that half-order fractional differential models describe the frequency dependence of the damping materials very well. Other authors have demonstrated applications of fractional differentials in the areas of non-Newtonian fluids [30], signal processing [31], viscoelasticity [32,33], fluid-dynamic traffic model [34], colored noise [35], bioengineering [36–38], solid mechanics [39], continuum and statistical mechanics [40], anomalous transport [41], economics [42].

Fractional differential equations have garnered a lot of attention and appreciation recently due to their ability to provide an exact description of different nonlinear phenomena. The process of development of models based on fractional-order differential systems has lately gained popularity in the investigation of dynamical systems. The advantage of fractional-order systems is that they allow greater degrees of freedom in the model. Recently, more and more investigators begin to study the qualitative properties and numerical solutions of fractional order biological models [43]. The main reason is that fractional-order equations are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. The authors of [44], explained the meaning of the fractional-order prey-predator model.

The organization of this paper is as follows. In the next section, we present preliminary results for our model. In Section 3, we present local stability of equilibrium. In Section 4 we present the numerical method. A brief discussion is given in Section 5 and 6.

2. Preliminaries

Definition 1. The Riemann–Liouville (R-L) fractional integral operator of order $\theta > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I^\theta f(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2. The Caputo fractional derivative of order $\theta > 0, n-1 < \theta < n, n \in \mathbb{N}$ is defined as

$$D^\theta f(t) = \frac{1}{\Gamma(n-\theta)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\theta-1} f(s)ds$$

and

$$D^\theta f(t) = \frac{1}{\Gamma(n-\theta)}\int_a^t \frac{f^{(n)}(s)}{(t-s)^{\theta+1-n}}ds,$$