Applied Mathematical Modelling 37 (2013) 8383–8393

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Contact problem for a thin elastic layer with variable thickness: Application to sensitivity analysis of articular contact mechanics

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ARTICLE INFO

Article history:
Received 27 April 2012
Received in revised form 13 March 2013
Accepted 24 March 2013
Available online 6 April 2013

Keywords:
Contact problem
Thin elastic layer
Variable thickness
Asymptotic model
Articular contact

ABSTRACT

In the framework of the recently developed asymptotic models for tibio-femoral contact incorporating frictionless elliptical contact interaction between thin elastic, viscoelastic, or biphasic cartilage layers, we apply an asymptotic modeling approach for analytical evaluating the sensitivity of crucial parameters in joint contact mechanics due to small variations in the thicknesses of the contacting cartilage layers. The four term asymptotic expansion for the normal displacement at the contact surface is explicitly derived, which recovers the corresponding solution obtained previously for the 2D case in the compressible case. It was found that to minimize the influence of the cartilage thickness non-uniformity on the force–displacement relationship, the effective thicknesses of articular layers should be determined from a special optimization criterion.

1. Introduction

Contact problems involving transmission of forces across biological joints are of considerable practical importance and a number of numerical models for articular contact are available [1,2]. At the same time, the necessity of analytical models becomes an important issue in developing improved understanding of load distribution in the normal and pathological joints, which affects the mechanical aspects of osteoarthritis [3,4]. Also, analytical modeling of the distributed internal forces generated by articular contact in tibio-femoral joints is required in multibody dynamic simulations of physical exercise of a human skeleton [5,6]. As a rule, analytical models of articular contact assume rigid bones and represent cartilage as a thin elastic layer of constant thickness resisting to deformation like a Winkler foundation consisting of a series of discrete springs with constant length and stiffness [7]. However, a subject-specific approach to articular contact mechanics requires developing patient-specific models for accurate predictions. Recently, a sensitivity analysis of finite element models of hip cartilage mechanics with respect to varying degrees of simplified geometry was performed in [8].

Based on the asymptotic analysis of the frictionless contact problem for a thin elastic layer bonded to a rigid substrate in the thin-layer limit [9,10], the following asymptotic model for contact interaction of two thin incompressible layers was established [11]:

$$-(E_1^{-1}h_1^3 + E_2^{-1}h_2^3)\Delta_y p(\mathbf{y}) = \delta_0 - \varphi(\mathbf{y}), \quad \mathbf{y} \in \omega, \tag{1}$$

$$p(\mathbf{y}) = 0, \quad \frac{\partial p}{\partial n}(\mathbf{y}) = 0, \quad \mathbf{y} \in \Gamma. \tag{2}$$

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http://dx.doi.org/10.1016/j.apm.2013.03.042

Here, $p(\mathbf{y})$ is the contact pressure density, h_α and E_α are the thickness and elastic modulus of the layer material, respectively, $\alpha = 1, 2, \Delta_y = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ is the Laplace differential operator, δ_0 is the vertical approach of the rigid substrates, $\varphi(\mathbf{y})$ is the gap function defined as the distance between the layer surfaces in the vertical direction, ω is the contact area, Γ is the contour of $\omega, \partial/\partial n$ is the normal derivative.

It was shown [12,3,13] that the problem (1) and (2) describes the instantaneous response of thin biphasic layers to dynamic and impact loading. In [14], the elastic model (1) and (2) was generalized for the general viscoelastic case.

With respect to articular contact, a special interest represents the case when the subchondral bones are shaped as an elliptic paraboloid

$$\varphi(\mathbf{y}) = (2R_1)^{-1} y_1^2 + (2R_2)^{-1} y_2^2, \tag{3}$$

with positive curvature radii R_1 and R_2.

In the case (3), the exact solution to the problem (1) and (2) has the following form [12,10]:

$$p(\mathbf{y}) = p_0 \left(1 - \frac{y_1^2}{a_1^2} - \frac{y_2^2}{a_2^2}\right)^2. \tag{4}$$

Integration of the pressure distribution (4) over the elliptical contact region ω with the semi-axes a_1 and a_2 results in the following force–displacement relationship [11]:

$$P = \frac{\pi m}{3} M_P(s) R_1 R_2 \delta_0^3. \tag{5}$$

Here, $M_P(s)$ is a dimensionless factor depending on the aspect ratio $s = a_2/a_1$, and the coefficient m is given by

$$m = (E_1^{-1} h_1^3 + E_2^{-1} h_2^3)^{-1}. \tag{6}$$

The asymptotic model (1)–(3) assumes that the cartilage layers have constant thicknesses, whereas it is well known [15] that articular cartilage has a variable thickness as well as the surface of subchondral bone deviates from the ellipsoid shape [16]. A sensitivity of the model (1) and (2) with respect to small perturbations of the gap function (3) was performed in [17]. In particular, it was shown [11] that the influence of the gap function variation on the force–displacement relationship will be negligible if the effective geometrical characteristics R_1 and R_2 are determined by a least square method.

To our knowledge, in the literature there is only one study [18] where the 2D case of contact problem for a thin elastic strip of variable thickness was solved by an asymptotic method under the assumption that Poisson's ratio of the strip material is not very close to 0.5. At the same time, many asymptotic solutions were derived for an elastic layer of constant thickness both in the axisymmetric [19–23] and the non-axisymmetric [24,9,10,25,26] cases. Contact problem for an incompressible elastic layer was studied in [20,27–29].

In the present paper, a three-dimensional unilateral contact problem for a thin elastic layer of variable thickness bonded to a rigid substrate is considered. Two cases are studied separately: (a) Poisson's ratio of the layer material is not very close to 0.5; (b) the layer material is incompressible with Poisson's ratio of 0.5. It is well known that the asymptotic solution for a thin elastic layer undergoes a dramatic change in the limit as $\nu \to 0.5$, so that the formulas obtained in the case (a) do not work when Poisson's ratio comes close to 0.5. After developing a refined asymptotic model, we apply sensitivity analysis to determine how "sensitive" is the mathematical model (1) and (2) to variations in the values of the layer thicknesses. To be more precise we consider the term "sensitivity" in a broad sense by allowing variable layer thicknesses, whereas the original model deals with scalar parameters h_1 and h_2.

2. Contact problem for a thin elastic layer with variable thickness

We consider a homogeneous, isotropic, linearly elastic layer with a plane contact surface, $x_3 = 0$, and a variable thickness, $H(x_1, x_2)$, firmly attached to an uneven rigid surface

$$x_3 = H(x_1, x_2). \tag{7}$$

In the absence of body forces, equations governing small deformations of the layer are

$$\frac{\partial \sigma_{1j}}{\partial x_1} + \frac{\partial \sigma_{2j}}{\partial x_2} + \frac{\partial \sigma_{3j}}{\partial x_3} = 0, \quad j = 1, 2, 3, \tag{8}$$

$$\sigma_{ij} = \lambda \delta_{ij}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{ij}, \tag{9}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{10}$$

where σ_{ij} is the Cauchy stress tensor, λ and μ are Lamé parameters for the layer material, ε_{ij} is the infinitesimal strain tensor, u_j is the displacement component along the x_j-axis, δ_{ij} is Kronecker's delta.