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Static snapping load of a hinged extensible elastica

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ABSTRACT

In this paper we discuss the static snapping load of a hinged buckled beam subject to a midpoint force. Three different models are compared; they are small-deformation theory, inextensible elastica, and extensible elastica. As expected, small-deformation theory fails to predict the static snapping load accurately in the large-deformation range. In the small-deformation range, on the other hand, inextensible elastica model predicts the static snapping load poorly. Finally, by allowing the elastica to be extensible, it is observed that one can predict the static snapping load accurately both in the small- and large-deformation ranges.

1. Introduction

An initially straight beam can be buckled into a curved shape by edge thrust. If both ends of the buckled beam are hinged in space, it becomes a natural bistable device. When the buckled beam is loaded laterally, the buckled beam may jump from one side to the other suddenly. This phenomenon is called snap-through buckling, which has wide applications in the design of bistable devices.

One of the goals of buckled beam research is to find the critical lateral load which initiates snap-through buckling. Two mathematical modeling methods are available in the current literature in this regard. In the first approach, small deformation is assumed and axial extensibility of the beam is permitted [1–6]. In this category, the snapping load may be obtained in closed form. In the second category, an elastica model taking into account exact geometry is adopted in the analysis of the deformation [7–10]. In this category, the beam is usually assumed to be inextensible. No closed-form snapping load is available in this approach. However, the restriction of small deformation is lifted.

One of the advantages of conventional elastica approach over small-deformation theory is that it can predict the snapping load even when the deformation of the buckled beam is large. However, the inextensible elastica model has its own problem, especially when the deformation is small. It will be shown in this paper that in the small-deformation range, the small-deformation theory works better than inextensible elastica theory. This stems from the fact that in conventional elastica theory the beam is assumed to be inextensible. In the small-deformation theory, on the other hand, the beam is allowed to be extensible, which is of course closer to the fact. In an effort to fix this defect of the conventional elastica theory, we propose in this paper to use extensible elastica to derive the snapping load of a buckled beam.

In 1972 Reissner [11] derived the equilibrium and strain–displacement equations for an extensible and shear-deformable planar elastica by using virtual work principle. Magnussen et al. [12] studied the buckling of an axially loaded beam and reported that the bifurcation point may change from supercritical (which is always the case for inextensible elastica) to subcritical when extensibility is considered in certain situations. The theory of extensible elastica has also been incorporated

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into finite element formulations [13,14]. Irschik and Gerstmayr [15] used continuum mechanics approach to formulate the equilibrium equations for an extensible, non-shear-deformable, and initially straight elastica. Humer and Irschik [16] studied the deformation of an extensible elastica resting between two spatially fixed supports with one end completely fixed and the other allowed to slide in the support after external load is applied.

In this paper we study the static snapping load of a hinged buckled beam under a midpoint force. The results from the three approaches, i.e., small-deformation theory, inextensible elastica, and extensible elastica, are compared. It will be shown that the extensible elastica theory not only can predict the snapping load accurately when the deformation is large, it can also produce consistent result in the small-deformation range.

2. Governing equations of an extensible elastica

We consider a uniform beam with total length L before deformation. The flexural rigidity of the elastic beam is EI. E is the Young's modulus and I is the area moment of inertia of the cross section. The effect of shear deformation is ignored. The elastic beam is straight initially when it is stress free. We assume that the beam is compressed by an axial force at the ends and buckles with the two ends being brought closer by a distance e^*. After this deformation, both ends O and B of the buckled beam are hinged in space, as shown in Fig. 1. The two dashed curves in Fig. 1 represent the two possible shapes of the buckled beam before any lateral load is applied. We fix an x^* y^*-coordinate system with its origin at point O. The buckled beam (elastica) is loaded at the midpoint $s^* = L/2$ by a point force Q^* pointing in the negative y^* -direction. s^* is the length of the undeformed beam measured from point O. The buckled beam may deform symmetrically or unsymmetrically. The solid curve in Fig. 1 represents the loaded beam when it deforms unsymmetrically under midpoint force Q^*. For a small element ds^*, one can write the geometrical relations [11],

$$\frac{\partial x^*(s^*)}{\partial s^*} = [1 + \varepsilon(s^*)]\cos\theta(s^*), \tag{1}$$

$$\frac{\partial y^*(s^*)}{\partial s^*} = [1 + \varepsilon(s^*)]\sin\theta(s^*). \tag{2}$$

$\theta(s^*)$ is the rotation angle of the neutral axis at s^*. $\varepsilon\,(s^*)$ is the relative change of length of the element ds^*,

$$\varepsilon(s^*) = \frac{1}{EA}[F_x^*(s^*)\cos\theta(s^*) + F_y^*(s^*)\sin\theta(s^*)]. \tag{3}$$

$F_x^*(s^*)$ and $F_y^*(s^*)$ are the internal forces in the x^*- and y^*-directions. The balance of moment can be written as

$$\frac{\partial M^*(s^*)}{\partial s^*} = [1 + \varepsilon(s^*)][F_x^*(s^*)\sin\theta(s^*) - F_y^*(s^*)\cos\theta(s^*)]. \tag{4}$$

$M^*(s^*)$ is the bending moment. It is assumed that the cross section of the beam remains plane and normal to the neutral axis after deformation. Shear deformation on the cross section plane is ignored. In other words, Euler–Bernoulli beam model is assumed. The relation between curvature and bending moment can be written as

$$\frac{\partial\theta(s^*)}{\partial s^*} = \frac{M^*(s^*)}{EI}. \tag{5}$$

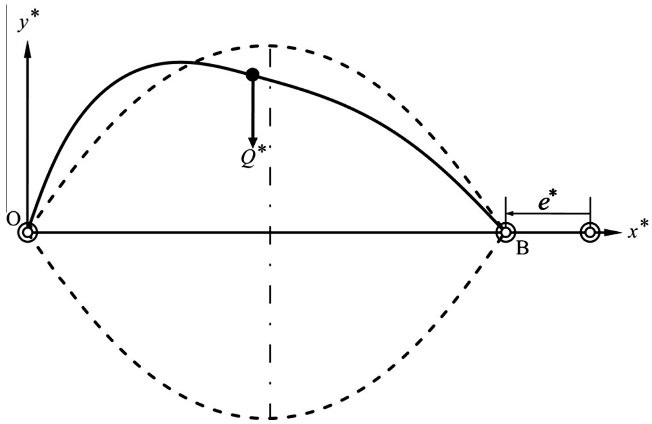

Fig. 1. A hinged buckled beam subject to a midpoint force.