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Numerical solution of the system of second-order boundary value problems using the local radial basis functions based differential quadrature collocation method

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ABSTRACT

In this research, we propose a numerical scheme to solve the system of second-order boundary value problems. In this way, we use the Local Radial Basis Function Differential Quadrature (LRBFDQ) method for approximating the derivative. The LRBFDQ method approximates the derivatives by Radial Basis Functions (RBFs) interpolation using a small set of nodes in the support domain of any node. So the new scheme needs much less computational work than the globally supported RBFs collocation method. We use two techniques presented by Bayona et al. (2011, 2012) [29,30] to determine the optimal shape parameter. Some examples are presented to demonstrate the accuracy and easy implementation of the new technique. The results of numerical experiments are compared with the analytical solution, finite difference (FD) method and some published methods to confirm the accuracy and efficiency of the new scheme presented in this paper.

1. Introduction

The system of second order boundary value problems is applied for modelling of several natural phenomena. Many of these problems are found in physics [1,2], engineering [3,4,2], biology [5] and so on. This system can be denoted by

$$\begin{cases} u'' + a_1(x)u' + a_2(x)u + a_3(x)v'' + a_4(x)v' + a_5(x)v + M_1(u, v, u', v') = f_1(x), \\ v'' + b_1(x)v' + b_2(x)v + b_3(x)u'' + b_4(x)u' + b_5(x)u + M_2(u, v, u', v') = f_2(x), \end{cases} \tag{1}$$

where M_1 and M_2 are nonlinear functions of u, v, u' and v', $f_1(x), f_2(x)$ and $a_j(x), b_j(x),\ j = 1, 2, 3, 4, 5$ are continuous functions. The boundary conditions are as

$$u(0) = A_1,\ \ u(1) = A_2,\ \ v(0) = B_1,\ \ v(1) = B_2, \tag{2}$$

where $A_i, B_i,\ i = 1, 2$, are constant.

The existence and uniqueness of solutions of system of second order boundary value problems (1), (2), including the approximation of solutions via finite difference equations, are addressed in [6–10].

For this nonlinear system of second-order BVPs, there are few valid methods for obtaining numerical solutions. Geng and Cui [11] represented the analytical solution of problem (1), (2) in the form of series in the reproducing kernel space and the

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approximate solution $u_n(x)$ was obtained from the n-term intercept of the analytical solution. Also, they recently used the homotopy perturbation-reproducing kernel method [12] and obtained better results than [11]. Lu [13] applied the variational iteration method to solve the nonlinear system of second-order boundary value problems (1), (2). Also Caglar et al. [14] studied the numerical solution of problem (1), (2) by B-spline method. Authors of [15–17] used the cubic B-spline, sinc-collocation and homotopy perturbation methods, respectively, and presented several test problems to demonstrate the validity and applicability of their techniques. We refer the interested reader to [17].

Our approach in the current paper is different. The Local Radial Basis Function Differential Quadrature (LRBFDQ) method proposed in [18] is utilized for the numerical solution of the system of second-order boundary value problems (1), (2). We discuss on both the linear and nonlinear cases of system (1), separately. When the system (1) contains nonlinear equations, we apply the Newton iteration technique [19] and show how the LRBFDQ method can be used for this case. The LRBFDQ method is a meshfree technique i.e, a set of scattered nodes is used instead of meshing the domain of the problem which saves the computational costs. Shu et al. [18] proposed this method to solve the two-dimensional incompressible Navier–Stokes equations. Wright [20] and Tolstykh and Shirobokov [21] introduced this method independently and approximately at the same time to solve the ODEs and elasticity problems, respectively. Of course, their method was RBF Finite Difference (RBF-FD) which is identical with LRBFDQ. So far this method has been used for the numerical solution of many problems, such as the boundary layer problems [22], the Sturm–Liouville problem [23], analysis of composite plates [24], the two-dimensional transient heat conduction problems [25], shallow water simulations on a sphere [26] and a solution to linear elasticity [27]. Also see [65,68,69,72]. In the LRBFDQ method, derivatives are approximated by RBF interpolation using a small set of nodes in the neighborhood of any node.

Discussion on the theoretical aspects of this method is found in [28–33]. Wright and Fornberg [33] showed that LRBFDQ method for flat RBF is equivalent to the standard finite difference scheme [34]. Bayona et al. [28] investigated the convergence properties of LRBFDQ and derived exact formulas to determine weight coefficients for the first and second derivatives in one dimension. Authors of [29,30] proposed algorithms to determine the optimal constant and variable shape parameter for Multiquadric function. Also they recently derived analytical expressions for the weights of Gaussian finite difference (GA-FD) and Gaussian Hermite finite difference (GA-HFD) formulas [31]. By these weights, they obtained analytical expressions for the leading order approximations to the local truncation error and showed that for each differential operator, there is a range of values of the shape parameter for which GA-FD and GA-HFD formulas are significantly more accurate than the corresponding standard FD formulas [31]. Also see [35,36].

Another research work on the RBF-FD method presented by Fornberg et al. [32]. In [37] they have been introduced stable algorithm RBF-QR. This algorithm covers unstability of the Globally Supported RBF (GSRBF) method for problems that their accuracy depends on very small shape parameter of RBF. Similar to that, they obtained a stable algorithm for computing the weight coefficients of RBF-FD method [32].

One of the advantages of our approach is its efficiency for system (1), (2) with the nonlinear terms as the functions of u, v, u' and v', while in the mentioned literature [11–17] the nonlinear terms of system (1), (2) are functions of u and v. The local property of this method overcomes the existing difficults in the following two classic methods:

(1) Globally Supported Radial Basis Function (GSRBF) collocation method
(2) Polynomial-based Differential Quadrature (PDQ) method.

The Globally Supported Radial Basis Function (GSRBF) collocation method proposed in [38,39] for the numerical solution of PDEs. Wu [40] described the advantages of using the globally RBFs as a meshless collocation method. These advantages are given in the following:

(a) This method does not require to generate mesh and is independent of the spatial dimension.
(b) Interpolation of scattered data with some RBFs has spectral convergence.

In spite of its simplicity and efficiency, the RBFs interpolation has a disadvantage that is the contradiction between its accuracy and stability [41]. As we know the condition number of the matrix of the RBFs interpolation becomes very large when the interpolation points are dense or irregularly spaced. Therefore when we use the meshless methods based on the radial basis functions (RBFs) [42,43,60,61,66,67,70,71] due to the globally supported RBF, we deal with a highly ill-conditioned dense matrix. We note that high ill-conditioning of the interpolation matrix will cause instability in computation. Moreover, using this method for solving the nonlinear or time-dependent problems increases the amount of computations. In order to improve these problems several strategies have been proposed in the literature [44–47,59,62,63,74] that one of them is the Localized RBF method. Therefore, in comparison with the GSRBF method, interpolation matrix of the LRBFDQ method is not ill-conditioned. Another additional benefit of this formulation is that we have much more freedom in choosing the shape parameter of the radial basis functions.

The classic form of the differential quadrature (DQ) method was introduced by Bellman et al. [48] for the numerical solution of PDEs [64,73,75–77] in 1972. This method is originated from the numerical integration approach and is a global technique. Civan and Sliepcevich [49] have shown that DQ method is inefficient when the number of grid points is large. In other words, using more grid points in DQ method causes more accurate results but are attached by the event of instability. To keep balance between the accuracy and stability, the Local Polynomial-based Differential Quadrature (LPDQ) method