Applied Mathematical Modelling 37 (2013) 1–12

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Recovering boundary data: The Cauchy Stokes system

Amel Ben Abda, Imen Ben Saad *, Maatoug Hassine

LAMSIN, Ecole Nationale d'Ingénieurs de Tunis, BP 37, 1002 Tunis belvédère, Tunisia

ARTICLE INFO

Article history:
Received 17 May 2011
Received in revised form 29 November 2011
Accepted 27 January 2012
Available online 22 February 2012

Keywords:
Cauchy–Stokes problem
Data completion
Interfacial equation
Inverse problem
Steklov–Poincaré operator

ABSTRACT

This paper is concerned with the severely ill-posed Cauchy–Stokes problem. We are interested in a data completion problem which is exploited to detect small leaks to control water loss Kim et al. (2008) [1]. This inverse problem is rephrased into an optimization one: An energy-like error functional is introduced. We prove that the optimality condition of the first order is equivalent to solving an interfacial equation which turns out to be a Cauchy-Steklov-Poincaré operator. Numerical trials highlight the efficiency of the method.
© 2012 Elsevier Inc. All rights reserved.

Contents

1. Introduction 1
2. Formulation of the problem 2
   2.1. Minimization problem 3
   2.2. The first order optimality condition 4
   2.3. The interfacial operators 6
      2.3.1. The Neumann-Dirichlet case: 6
      2.3.2. The Dirichlet–Dirichlet case: 6
      2.3.3. The Neuman–Neuman case: 7
3. The numerical procedure 7
4. Comparison with Kozlov–Maz'ya–Fomin's algorithm 7
4. Comparison with Kozlov–Maz'ya–Fomin's algorithm 8
5. Numerical illustration 9
6. Conclusion 12
References 12

1. Introduction

Let $\Omega \subset \mathbf{R}^d,\ d = 2,3$, be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. We assume that Γ is partitioned into two parts Γ_a and Γ_i having both non-vanishing measure.

* Corresponding author.
E-mail addresses: Amel.benabda@enit.rnu.tn (A.B. Abda), imen.bensaad@lamsin.rnu.tn (I.B. Saad), maatoug.hassine@enit.rnu.tn (M. Hassine).

0307-904X/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.apm.2012.01.055

In this work we are interested in a data completion problem related to the Stokes system. It consists in recovering the data on the incomplete (inaccessible,...) boundary Γ_i from the over-specified data on the accessible boundary Γ_a. As application of this problem is the leaks detection which can be useful for the losses of water [1].

Assume a given velocity U and a force F on Γ_a, the data completion problem for the Stokes operator can be formulated as a Cauchy problem type: *find the velocity field u and the pressure p solution to*

$$\begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla . u = 0 & \text{in } \Omega, \\ u = U & \text{on } \Gamma_a, \\ \sigma(u).n = F & \text{on } \Gamma_a, \end{cases} \tag{1}$$

where ν is the fluid kinematic viscosity and σ is the stress tensor $\sigma(u) = 2\nu D(u) - pI$, $D(u)$ denotes the deformation tensor $D(u) = 1/2(\nabla u + \nabla u^T)$, I is the $d \times d$ identity matrix and n is the unit outward normal vector. This problem is known since Hadamard to be ill-posed in the sense that the dependence of (u,p) on the data (U,F) is not continuous. In order to reconstruct the unknown boundary data $u_{|\Gamma_i}$ and $\sigma(u).n_{|\Gamma_i}$ on Γ_i we will use the Steklov–Poincaré operator (see [2] or [3] or [4] for the Laplace equation). The inverse problem is formulated as an optimization one, the first optimality condition gives rise to an interfacial equation involving the Dirichlet-to-Neumann operator. There is very little litterature dealing with the Stokes–Cauchy problems. We would like to mention the work [5] where the data recovering process reads as a least square tracking of the given data. We will also refer to the alternating iterative algorithm in [6,7] for elliptic equations and in [8–12] for the stationary Stokes system. This paper is outlined as follows: The next Section is devoted to the formulation of the Cauchy problem for the Stokes system. The compatibility data notion is discussed. We recall that the set of compatible data is dense on $H^{1/2}(\Gamma_a)^d \times H^{-1/2}(\Gamma_a)^d$. An energy-like error functional is introduced in the context of the ill-posed problem of recovering boundary data. In Section 2.1, the data completion problem is formulated as an optimization one. In Section 2.2, the first order optimality condition is rephrased in terms of an interfacial problem using the Steklov–Poincaré operator [13,14]. The numerical procedure for solving the Stokes–Cauchy problem is described in Section 3. The Kozlov–Maz'ya–Fomin algorithm (the KMF algorithm) is adapted for the Stokes system in Section 4. Section 5 is devoted to some numerical illustrations to compare the proposed method for the data recovering problem with the KMF algorithm. The closing section is devoted to comments.

2. Formulation of the problem

Let us consider the above Cauchy problem (1). Assume that the data (U,F) are "compatible", i.e. that is this pair is indeed the trace and stress tensors of a unique function (u,p). Extending the data means finding (V,G) such that:

$$\begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla . u = 0 & \text{in } \Omega, \\ u = U, \sigma(u).n = F & \text{on } \Gamma_a, \\ u = V, \sigma(u).n = G & \text{on } \Gamma_i. \end{cases} \tag{2}$$

The question is to reconstruct numerically the pair (V,G), on the inaccessible boundary Γ_i. However, all the results stated are also true in the case of less smooth boundaries and when Γ_a and Γ_i have contact points. It is proven in [3,15], for the Laplace equation, that the pairs of compatible data (U,F) are dense in $H^{1/2}(\Gamma_a)^d \times H^{-1/2}(\Gamma_a)^d$. In the following lemma we establish that the same density result can be easily extended for the Stokes Cauchy problem.

Lemma 2.1. *Let (U,F) be a given data.*

1. *For a fixed U in $H^{1/2}(\Gamma_a)^d$, the set of data F for which there exists (u,p) in $H^1(\Omega)^d \times L^2(\Omega)$, satisfying the Cauchy problem (1) is everywhere dense in $H^{-1/2}(\Gamma_a)^d$.*
2. *For a fixed F in $H^{-1/2}(\Gamma_a)^d$, the set of data U for which there exists (u,p) in $H^1(\Omega)^d \times L^2(\Omega)$, satisfying the Cauchy problem (1) is everywhere dense in $H^{1/2}(\Gamma_a)^d$.*

Proof. Let us prove the first assertion, the second one can be obtained by the same arguments. It is sufficient to prove the result for $U = 0$. Let (u,p) be the solution to the problem:

$$\begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla . u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_a, \\ \sigma(u).n = F & \text{on } \Gamma_a. \end{cases} \tag{3}$$

Assume, now, that the first assertion fails. We denote by R the set of the compatible data F with the Dirichlet condition $U = 0$ on Γ_a,

$$R = \{F \in H^{-1/2}(\Gamma_a)^d, \quad \text{such that } (0,F) \quad \text{is a compatible data}\},$$

Download English Version:

https://daneshyari.com/en/article/8052995

Download Persian Version:

https://daneshyari.com/article/8052995

Daneshyari.com