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Fast synchronization of directionally coupled chaotic systems

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This paper studies the fast synchronization of directionally coupled chaotic systems under a chained interaction topology. Firstly, by applying finite-time stability theory, it is shown that all chaotic systems can achieve synchronization in finite time as long as the coupling strength is strong enough. Secondly, it is proved that the settling times are determined by the interaction strength, system parameters and initial conditions of the chaotic systems. Furthermore, it is found that the settling times are mainly dependent on the bounded value and dimension of the coupled chaotic systems when the individual chaotic sub-system is bounded. Finally, illustrative examples and numerical simulations are given to show the correctness of theoretical results.

1. Introduction

Over the past several years, synchronization of coupled chaotic systems (oscillators) has received much attention due to its wide application in secure communication, image and information processing, complex networks and biological systems. Synchronization of two identical chaotic systems with different initial conditions was first studied by Pecora and Carroll [1]. Various methods for synchronization of chaotic systems have been proposed, such as linear state feedback control [2–7], adaptive control and backstepping nonlinear control [8–13], hybrid control method [14], passivity-based control and sliding mode control [15–17] (see the references therein).

Recent studies of chaos synchronization have more focused on the convergence rate of coupled chaotic systems. The main drive for studying the convergence rate is that from a practical point of view the coupled chaotic systems are required reaching synchronization within a short period of time. By using above mentioned conventional control methods, the controlled chaotic systems can only achieve asymptotic convergence and their trajectories reach synchronization in infinite time. In order to achieve fast synchronization, an effective method is the use of finite-time control techniques to optimize convergence time in coupled systems [18–35]. The finite-time control techniques have demonstrated better robustness, disturbance rejection and fast convergence rate properties [18,19]. Other methods regarding the finite-time stability of autonomous and time-varying systems, such as pure finite-time control [20–23], adaptive finite-time and linear state feedback control [24–27], sliding mode control method [28–30], output feedback control [31,32], finite-time stochastic control method [33] and CLF-based control method [34,35] have also been developed.

This paper investigates the fast synchronization of directionally coupled chaotic systems in a chained interaction topology. The directed topology is considered based on the fact that most of informations exchange between oscillators are directed, in other words, the coupled oscillators in a group cannot receive information from each other and only receive or send

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information to their neighbours. Unlike the studies of coupled oscillators with undirected topology, the finite-time synchronization and its criteria of coupled oscillators in directed network topology have so far received little attention in the literatures. It is thus necessary to consider the directionally coupled chaotic systems and derive the finite-time convergence conditions. The methods to be developed for chained coupled systems can provide preliminary results on how to select the Lyapunov function and derive the convergence criteria of multiple chaotic systems in a complex directed network topology.

The main aim of this paper is to develop the convergence conditions and settling times for the directionally coupled chaotic systems. Specifically, this paper studies under what conditions such coupled systems can achieve synchronization and how quickly they can reach finite-time synchronization. Furthermore, synchronization of the coupled system containing bounded individual chaotic sub-systems will also be investigated. This consideration is based on the fact that the exactly initial conditions may not be easily obtained in some cases (such as the random initial conditions), instead, the bounded values of all individual chaotic systems can be estimated by using the physical devices limitation or the experimental results. When the individual chaotic system is bounded, simpler controllers can be used to ensure such coupled systems to achieve finite-time synchronization. It will be shown that the settling times are mainly determined by the bounded value and dimension of all chaotic systems for given interaction strength and system function parameter.

The rest of this paper is organized as follows. In Section 2, the notations and problem formulations are presented. Section 3 provides the main results of finite-time synchronization without and with linear interactions. Illustrative examples and numerical simulations are presented in Section 4. Conclusion for the coupled systems is given in Section 5.

2. Problem formulations

Throughout this paper, let R define a set of real numbers; R^n be the n-dimensional real vector; $R^{n\times n}$ be the set of $n \times n$ real matrices; $diag(a_1,\ldots,a_n)$ denote a diagonal matrix with diagonal elements $a_1,\ldots,a_n$; the sign matrix function $sign(x_j(t)-x_i(t)) = diag(sign(x_{j1}(t)-x_{i1}(t)),\ldots,sign(x_{jn}(t)-x_{in}(t))) \in R^{n\times n}$, $|x_j(t)-x_i(t)|^{\theta} = (|x_{j1}(t)-x_{i1}(t)|^{\theta},\ldots,|x_{jn}(t)-x_{in}(t)|^{\theta})^{\top} \in R^n$, $sign(x_j(t)-x_i(t))^{\theta} = sign(x_j(t)-x_i(t))|x_j(t)-x_i(t)|^{\theta}$, $0<\theta<1$ and $sign(z)(z\in R)$ be the sign function.

The equations of directionally coupled chaotic systems in a chained network topology can be described by the following differential equations [21]:

$$\dot{x}_i(t) = f(x_i(t)) + c_{i,i+1}sign(x_{i+1}(t)-x_i(t))^{\theta} + c_{i,i-1}sign(x_{i-1}(t)-x_i(t))^{\theta} + d_{i,i+1}(x_{i+1}(t)-x_i(t)) + d_{i,i-1}(x_{i-1}(t)-x_i(t)),\quad i=1,\ldots,N. \tag{1}$$

Here, $x_i(t) = (x_{i1}(t), x_{i2}(t),\ldots,x_{in}(t))^{\top} \in R^n$ is the state vector of system i. $c_{i,i+1}$ and $c_{i,i-1}$ are the coupled nonlinear strengths of system i with its neighbour systems $i+1$ and $i-1$, respectively. If system i can send information to system $i+1$, then $c_{i+1,i}>0$, otherwise $c_{i+1,i}=0$. Similarly, $d_{i,i-1}$ and $d_{i,i+1}$ are the coupled linear strengths of the system. $f: R^n \to R^n$ is Hölder continuous vector-valued function satisfying $f(0)=0$. There are some smooth functions, such as $\sin(z)$, $\ln(1+z^2)$ and $\arctan(z)(z\in R)$ [32] satisfying this property. Since the network topology is chained, any sub-system in the network has two neighbours except the first and last one. In order to ensure that the first and last sub-system each has one neighbour only, it is assumed that $c_{10}=d_{10}=c_{N,N+1}=d_{N,N+1}=0$ in system (1). Such an assumption makes system (1) mathematically meaningful for $i=1,\ldots,N$. Here c_{10} and d_{10} can be considered as the coupling strengths of sub-system 1 with its virtual neighbour, sub-system 0. While for sub-system N, $c_{N,N+1}$ and $d_{N,N+1}$ can be regarded as the coupling strengths of sub-system N with its virtual neighbour, sub-system $N+1$. In reality, for a system composed of N sub-systems, sub-system 0 and sub-system $N+1$ do not exist.

In practical applications, it is hoped that simple control inputs can be used to guarantee all oscillators in system (1) to attain finite-time synchronization. However, in most cases the linear and non-linear control inputs in (1) are necessary for general systems of coupled chaotic oscillators to attain synchronization in finite time. When the oscillator (or sub-system) is bounded, the obtained results show that all oscillators in (1) can attain fast synchronization by only using the non-linear control input (i.e., letting the linear coupling $d_{i,i+1}=d_{i,i-1}=0$ in Eq. (1)). So, system (1) can be synchronized by using simple controllers, which can be rewritten as:

$$\dot{x}_i(t) = f(x_i(t)) + c_{i,i+1}sign(x_{i+1}(t)-x_i(t))^{\theta} + c_{i,i-1}sign(x_{i-1}(t)-x_i(t))^{\theta},\quad i=1,\ldots,N. \tag{2}$$

In most physically realizable systems and coupled oscillators, such as Lorenz systems, Rösystems, Genesio system, Chen systems, chaotic motors, harmonic oscillators, Chua's circuits and Duffing oscillators, it was found that all trajectories of such chaotic oscillators stay in a bounded region at all times [25,36]. When the individual chaotic system is bounded, the theoretical results show that all coupled oscillators given in (2) can achieve fast synchronization and the settling time is mainly dependent on the bounded value and dimension of coupled systems.

System (2) is simpler than (1) since it does not have linear interactions between the chaotic systems. In this paper, it is always assumed that systems (1) and (2) have a unique solution in forward time with respect to initial conditions [20]. The objective of the present paper is to derive the conditions such that $\lim_{t\to T_0^-}(x_{jl}(t)-x_{il}(t))=0$ for all oscillators in systems (1) and (2) $(i,j=1,\ldots,N,\ l=1,\ldots,n)$, where $t\to T_0^-$ denotes t approaches T_0 (settling time) from the left. In other words, systems (1) and (2) can achieve fast (finite-time) synchronization under given convergence conditions.