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High accuracy cubic spline approximation for two dimensional quasi-linear elliptic boundary value problems

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ABSTRACT

We report a new 9 point compact discretization of order two in *y*- and order four in *x*-directions, based on cubic spline approximation, for the solution of two dimensional quasi-linear elliptic partial differential equations. We describe the complete derivation procedure of the method in details and also discuss how our discretization is able to handle Poisson's equation in polar coordinates. The convergence analysis of the proposed cubic spline approximation for the nonlinear elliptic equation is discussed in details and we have shown under appropriate conditions the proposed method converges. Some physical examples and their numerical results are provided to justify the advantages of the proposed method.

1. Introduction

When a physical system depends on more than one variable a general description of its behavior often leads to partial differential equation. These equations arise in such diverse subjects as meteorology, electromagnetic theory, heat transfer, nuclear physics and elasticity to name just a few. Often, a system of differential equations of various order with different boundary conditions occur in the study of obstacle, unilateral, moving and free boundary value problems, problems of deflection of beams and in a number of other scientific applications. Most of these equations arising in applications are not solvable analytically and one is obliged to devise techniques for the determination of approximate solutions.

In the present paper, we aim to discuss the application of cubic spline functions to solve these boundary value problems. Use of cubic spline functions in the solution of nonlinear boundary value problems has been a challenging task for academic researchers. For the two dimensional linear elliptic equations, a number of constant mesh fourth order finite difference schemes have been designed by [1–5]. These linear systems have good numerical stability and provide high accuracy approximations. Later, using 9 point fourth order discretization for the solution of two dimensional non linear boundary value problems have been developed by Jain et al. [6,7], Mohanty [8], Mohanty and Singh [9] and Mohanty et al. [10]. Theory of splines and their applications to two point linear boundary value problems have been studied in [11–17]. Jain and Aziz [18] have derived fourth order cubic spline method for solving nonlinear two point boundary value problems with significant first derivative terms. Al-Said [19,20] has used cubic splines in the numerical solution of the second order boundary value problems. In the recent past, Mohanty et al. [21,22] have discussed fourth order accurate cubic spline Alternate Group Explicit

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method for the solution of two point boundary value problems. Khan et al. [23,24] have analyzed the use of parametric cubic spline for the solution of linear two point boundary value problems. Later, Rashidinia et al. [25,26] have derived the cubic spline method for the nonlinear singular two point boundary value problems. Houstis et al. [27] have first used point iterative cubic spline collocation method for the solution of linear elliptic equations. Later, Hadjidimos et al. [28] have extended their technique and used line iterative cubic spline collocation method for the solution of elliptic partial differential equation. Recently, Mohanty et al. [29–32] have derived high accuracy finite difference methods for the numerical solution of non-linear elliptic, hyperbolic and parabolic partial differential equations. To the author's knowledge, no high order cubic spline method for the solution of two dimensional quasi linear elliptic partial differential equations is known in the literature so far.

We now develop a new high accuracy 9-point (see Fig. 1) cubic spline discretization for the solution of two dimensional quasi-linear elliptic partial differential equation of the form:

$$A(x,y,u)\frac{\partial^2 u}{\partial x^2}+B(x,y,u)\frac{\partial^2 u}{\partial y^2}=f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right),\quad (x,y)\in\Omega \tag{1}$$

defined in the domain $\Omega = \{(x,y): 0 < x,y < 1\}$ with boundary $\partial\Omega$, where $A(x,y,u) > 0$ and $B(x,y,u) > 0$ in Ω. The value of u is being given on the boundary of the solution domain Ω.

The corresponding Dirichlet boundary conditions are prescribed by

$$u(x,y)=\psi(x,y),(x,y)\in\partial\Omega. \tag{2}$$

We assume that for $0 < x,\ y < 1$,

(i) $f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$ is continuous,

(ii) $\frac{\partial f}{\partial u},\frac{\partial f}{\partial u_x},\frac{\partial f}{\partial u_y}$ exist and are continuous,

(iii) $\frac{\partial f}{\partial u}\geqslant 0, \left|\frac{\partial f}{\partial u_x}\right|\leqslant G$ and $\left|\frac{\partial f}{\partial u_y}\right|\leqslant H$

where G and H are positive constants (see [6,7]). Further, we may also assume that the coefficients $A(x,y,u)$ and $B(x,y,u)$ are sufficiently smooth and their required higher order partial derivatives exist in the solution domain Ω.

The main aim of this work is to use cubic spline functions and their certain consistency relations, which are then used to develop a numerical method for computing smooth approximations to the solution of Eq. (1). Note that each discretization of the elliptic differential Eq. (1) at an interior grid point is based on just 3 evaluations of the function f. The paper is designed as follows. The next section of this paper presents the high accuracy numerical method based on cubic spline approximations. Third section discusses the derivation procedure of the scheme developed. Section 4 concerns with establishing the convergence analysis of the proposed method. In Section 5, we discuss the application of proposed method to polar coordinate problems. Section 6 gives the results of the numerical experiments to verify the accuracy and computational efficiency of the proposed method. Finally, the paper concludes with some brief remarks on the present work.

2. The cubic spline approximation and numerical scheme

In this section, we first aim to discuss a numerical method based on cubic spline approximation for the solution of non-linear elliptic equation

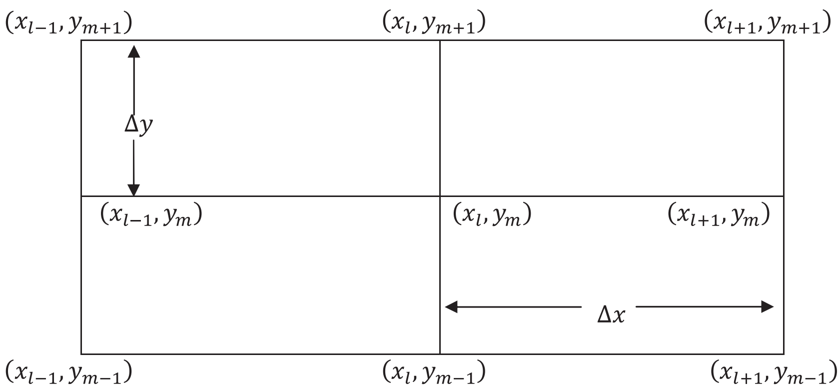

Fig. 1. 9-Point computational network.