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Analytical solutions for heat transfer on fractal and pre-fractal domains

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ABSTRACT

Fractals can be used to represent intricate self-similar geometries, but their application to the representation of physical systems is beset with difficulties which stem from an inability to define traditionally derived-physical quantities such as stress, pressure, strain, heat etc. This paper describes a method for the determination of analytical heat-transfer solutions on pre-fractal and fractal domains. The approach requires the construction of maps from pre-fractal domains to the continuum, which facilitate the application of traditional continuum solution methods. Solutions on fractal domains are achievable with this approach, and are defined to be the limit solution of analytical solutions obtained on the pre-fractals approximating the fractal of interest. This approach avoids many of the complications and technical difficulties arising from the use of measure theory and fractional derivatives, but also infers that the governing heat transfer equations are valid on all pre-fractals. The fractals considered are necessarily deterministic and relatively simple in form to demonstrate the solution methodology. The solutions presented are limited to one and two-dimensional domains and, in 1-D, are applied to an idealised composite material consisting of relatively small particles of infinitely low thermal conductivity embedded in a relatively large matrix of infinitely high thermal conductivity. The fractal composite system is thus not truly representative of a realistic physical system, but the methods presented do serve to demonstrate how analytical solutions can be attained on dust-like fractal domains. It is demonstrated that a measurable temperature is possible on a fractal structure along with finite measures of heat flux and energy. Transient and steady state thermal solutions are presented. The solutions on a selection of the pre-fractals are compared against finite element predictions to reinforce the validity of the approach.

1. Introduction

Conductive heat transfer on a continuum body is reasonably well described by Fourier's law of heat conduction. The physics underpinning thermal conduction is assumed to be scale independent on a continuum. Quantities of particular interest in conduction heat transfer are temperature, internal energy, enthalpy, thermal capacity, thermal conductivity and density. The principal link between the material and spatial volume is the density. In continuum thermodynamics, density is defined at a point by considering the ratio of material mass to volume in the limit as the volume shrinks to zero. It is recognised that this limit breaks down at molecular scales and at meso-length scales for some materials. The continuum property provides a route for a heterogeneous microstructure to an essentially homogeneous continuum. However, it has become necessary to develop complex material models and the associated physics in order to more accurately represent the behaviour of complex heterogeneous structures present at the macro and meso-scales with modern composites and cellular structures.

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The focus in this paper is on the construction of analytical solutions for heat transfer on dust-like fractals and their associated pre-fractals. It is unfortunate that, unlike continuum thermodynamics, there presently exist no analytical solutions for heat transfer on fractal-like microstructures. Analytical solutions in a continuum setting provide the foundation for the development of complex numerical procedures as these are used for essential validation. The view adopted in this paper is that the absence of analytical solutions in fractal thermodynamics is severely impeding the development of the subject.

Consider for example the theoretical work of work of Tarasov [1,2] and more recently Ostraja-Starzewski [3–5] with the development of continuum-like transport equations for mass, momentum and energy on fractal porous structures. Their approach (like others [6–18]) is founded on the application of fractional derivatives. Most theories are untested and physically unrealisable in many respects. An associated problem is that there is presently no unified accepted definition for a fractional derivative as each has deficiencies, although the one discovered by Liouville and extended by Riemann is commonly used [17]. Other commonly used derivatives are the Grunwald–Letnikov derivative and the Caputo derivative which are similar in form to the Riemann–Liouville derivative [18].

An alternative method that avoids fractional derivatives is via the indirect use of fractal quantities. One such quantity is the fractal dimension of some representative fractal applied to a continuum type model, but such approaches have severe limitations and are not considered further here [19,20].

Aside from generic theories, research into particular solution methods has been developed in an attempt to cater for heat transfer through irregular structures. In Ref. [21] for example, a modified Schwarz-Christoffel transformation and an integral representation is used to represent heat transfer through an irregular surface. The work is numerical and limited to pre-fractals, and no analytical results are presented. An alternative numerical approach performed in Ref. [22] is the use of the lattice Boltzmann model to simulate fluid-solid coupling and heat transfer. This approach has the advantage of having clear physical meanings attached to the procedure, although again it is numerical and limited to pre-fractals.

The methodology presented in this paper is founded on the existence of mappings between pre-fractals and the continuum. Governing partial differential equations (or transport equations) applicable to the pre-fractal can then be related to similar scaled equations on the continuum. In two and three dimensions the work thus far is limited to product fractals, but these do serve to demonstrate how geometric anisotropy at the pre-fractal scale manifests itself as material anisotropy at the continuum scale. Use can then be made of existing continuum solutions to generate analytical and approximate solutions on the pre-fractal. In fact, in this way, decades of work performed in continuum theories can be applied to pre-fractals and (in the limit) fractals, and hence establish a solid foundation for heat transfer on fractals and pre-fractals which has hitherto been lacking. Although the approach is generic, it does rely on the existence of particular mappings between pre-fractals and the continuum which can be readily shown to exist for all dusts and product spaces formed by dusts, which are the focus of this paper. Once maps have been established, the solution method is straightforward and simply requires a change in the spatial arguments of the continuum solution to produce pre-fractal solutions and limiting fractal solutions. The weakness in the approach currently is the lack of generic mappings, which limits the scheme to relatively simple fractals. However, despite this, the ability to generate analytical solutions is considered by the authors to be critical for the sound development of the subject area.

The composite fractal-material model on which calculations are performed is presented in Section 2, and consists of infinitely-thin, high-density, low-conducting particles embedded in infinitely-low density conducting matrix material. The overall composite possesses an average thermal conductivity, and in one dimension the fibres are assumed to be spatially distributed along a rod as a Cantor dust. In two and three dimensions, product fractals are used to describe the distribution of the composite fibres. One of the downsides of obtaining solutions on a fractal is that material models by necessity are extreme in nature, possessing a mix of material properties that is typically infinite or zero. This is, of course, not the case on pre-fractals which are much more useful for representing physical systems.

The Cantor dust and associated contraction mappings are introduced in Section 3 along with mappings from the pre-fractals to the continuum. The steady state solution arising from the uni-axial heat loading of the composite rod, (i.e. the Devil's staircase) is presented in Section 4 to introduce the concept of an equivalent-continuum problem. It is demonstrated through this concept just how readily steady-state solutions on a fractal rod (constructed from an idealised-composite material) can be obtained for a range of thermal loading conditions. The authors are unaware of any other steady-state analytical solution beyond the Devil's staircase. Transient heat transfer problems are presented in Section 5. The method presented is generic in the sense that it shows how continuum transient solutions can be used to obtain solutions on a fractal composite. In Section 6 solutions on pre-fractals are contrasted against solutions obtained using the finite element method to provide confidence in the physical representations. In Sections 7 and 8 the methods are extended to product fractals, where analytical steady state and transient temperatures are obtained for isotropic and anisotropic fractal structures.

2. The 1-D composite fractal material model

In order to demonstrate the method and to obtain analytical solutions, an idealised-composite material is used. Attention is limited to a binary composite consisting of two isotropic materials with possibly extreme thermal properties. In 1-D, the composite considered consists of infinitely small particles possessing an infinitely small value for thermal conductivity, separated by differing sizes of layers of infinitely high conductivity material. Under heat loading, such a material will sustain high temperature gradients in the particles and low temperature gradients in the matrix. The steady heat loading of a fractal