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Modulation of stochastic diffusion by wave motion

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ABSTRACT

Pollutants that are chemically inert flow with the carrier fluid passively while diffuse at the same time. In this study, the stochastic diffusion behavior of the passive pollutant in a progressive or standing wave field is examined with analytical means. Our focus is on the nonlinear interactions between the stochastic diffusion and the deterministic wave motions, and we limit the scope to cases whereby a small parameter, ε, exists between the advective and diffusive displacements, which then allows a perturbation analysis to be performed. With a sinusoidal progressive wave, the results show that the deterministic wave motion can either increase or decrease the embedded stochastic diffusion depending on the wave characteristics. Longer wave lengths and shorter wave periods tend to promote diffusion significantly, while shorter wave lengths and longer wave periods act in the opposite manner but with a much smaller effect. An analysis of the standing wave motion, represented by a combination of left and right moving progressive waves, shows that the effects due to two opposing waves to the stochastic diffusion can be superimposed.

1. Introduction

Pollutants that are chemically inert flow with the carrier fluid passively while diffuse at the same time. Over the years, a good level of understanding has been accumulated on the transport of these passive pollutants in a steady flow field (e.g. [1]). The steady analysis is typically based on the advection–dispersion equation with the mean velocity transporting the pollutants downstream, while the molecular and/or turbulence dispersion act to reduce the concentration peak while enlarging the Gaussian width. The analysis also typically assumes no direct link between the advective and dispersive displacements, such that they can be decoupled for analysis by means of a Galilean transformation of the coordinate with the mean flow velocity. A well-known example with this approach is the quantitative analysis for the diffusion in steady pipe flows, which results can be found in many textbooks, e.g. [1].

Compared to steady flows, the corresponding understandings of the diffusion in an oscillating flow field are not developed as well. This is despite the fact that the general knowledge of the diffusion in oscillation flows covers many physical phenomena. Some prior studies include Watson [2] who investigated the diffusion phenomenon in oscillatory pipe flows and evaluated the diffusion coefficient. Mei et al. [3] examined the transport and resuspension of fine particles in an oscillatory tidal boundary near a small peninsula. Hazra et al. [4] analyzed the solute dispersion in a channel during a periodic oscillating flow, and pointed out that the longitudinal dispersion due to molecular diffusion and non-uniform cross-section velocity can be influenced by the frequency and amplitude of the flow. Recently, Law [5] and Huang and Law [6] examined the longitudinal dispersion induced by Stokes drifts under monochromatic and random waves, respectively. The above studies again generally focus on conditions whereby the advective and diffusive displacements of the pollutant can be decoupled, in an approximate manner akin to the Generalized Lagrangian Mean theory [7].

The Eulerian wave-induced oscillatory flow field, despite its deterministic nature, can in fact influence the diffusive behavior depending on the wave characteristics. This situation is rarely addressed. One exception is Jansons [8] who explored an alternative method to calculate the Taylor dispersivity in oscillatory flows and obtained exact expressions for some specific cases. In this study, we examine the stochastic diffusion of passive substances in a progressive and standing wave field, with the focus on the nonlinear interactions between the stochastic diffusive behavior and the deterministic oscillatory advective motion. The random Brownian diffusion, while exact in simulating the molecular motion, can also provide a basic representation of the dispersion induced by ambient turbulence that results from residual vorticity generated by flow separation from boundaries away from the region and transported to the flow domain. In this study, we limit our scope to an environment whereby a small parameter ε exists between the advective and the diffusive motion. This allows a perturbation analysis to be performed, leading to analytical expressions that provide insights to the interaction.

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2. Analysis

In a wavy environment with embedded diffusion, the incremental change in the particle trajectory X_t is assumed to be governed by the following stochastic differential equation:

$$\mathrm{d}X_t = f(X_t, t)\,\mathrm{d}t + \mathrm{d}W_t \tag{1}$$

where f is the orbital velocity in an oscillatory flow, W_t is the Brownian motion with $\mathrm{d}W_t \sim N(0, \sigma^2\mathrm{d}t)$, i.e., Gaussian with zero mean and variance $\sigma^2\mathrm{d}t$; $\frac{\sigma^2}{2}$ is a diffusion constant. The above formulation was used by Jansons and Lythe [9] to investigate the kinematics of particle motion, with the objective of finding the resulting Stokes drift in the stochastic flow. To facilitate the analysis, we first define the non-dimensional parameters as follows:

$$X_t \to \frac{X_t}{k}; \quad t \to \frac{t}{\omega}; \qquad \sigma \to \frac{\sigma\sqrt{\omega}}{k}; \qquad f \to fA \tag{2}$$

where $k = 2\pi/L$ is the characteristic wave number and L is the characteristic wave length; $\omega = 2\pi/T$ is the characteristic wave angular frequency and T is the characteristic wave period; A is the characteristic amplitude of the wave-induced velocity. The governing equation is then transformed to

$$\mathrm{d}X_t = \varepsilon f(X_t, t)\,\mathrm{d}t + \mathrm{d}W_t \tag{3}$$

where $\varepsilon = kA/\omega$, and $\mathrm{d}W_t \sim N\left(0, \sigma^2\mathrm{d}t\right)$.

Consider a perturbation series of X_t for the small parameter ε:

$$X_t = X_t^{(0)} + \varepsilon X_t^{(1)} + \varepsilon^2 X_t^{(2)} + O\left(\varepsilon^3\right). \tag{4}$$

Jansons and Lythe [9] showed that the zeroth order motion is caused directly by the Brownian motion, and the first order motion is driven by the oscillatory flow, i.e.,

$$X_t^{(0)} = W_t \quad \text{and} \quad X_t^{(1)} = \int_0^t f\left(W_s, s\right) \mathrm{d}s. \tag{5}$$

We now proceed to analyze the diffusion behavior of the passive substance. Writing (3) in an incremental form as

$$X_{t+\Delta t} = X_t + \varepsilon f(X_t, t)\Delta t + \Delta W_t \tag{6}$$

where $\Delta W_t = W_{t+\Delta t} - W_t$, and taking variance on both sides and noting that ΔW_t is independent of the other two terms on the RHS, one obtains the diffusion behavior after rearranging and taking $\Delta t \to 0$ as

$$\frac{\mathrm{dVar}\,X_t}{\mathrm{d}t} = 2\varepsilon \operatorname{cov}\left(X_t, f\left(X_t, t\right)\right) + \sigma^2. \tag{7}$$

We examine the large time behavior of Var X_t, and hence of cov(X_t, $f(X_t, t)$). By definition,

$$\operatorname{cov}(X_t, f(X_t, t)) = \langle X_t f(X_t, t)\rangle - \langle X_t\rangle \langle f(X_t, t)\rangle \tag{8}$$

where $\langle . \rangle$ denotes the ensemble average or mathematical expectation. Taking expectation on the integrated form of (3) and noting that $\langle \mathrm{d}W_t\rangle = 0$, one obtains

$$\langle X_t\rangle = \varepsilon \int_0^t \langle f(X_s, s)\rangle \,\mathrm{d}s. \tag{9}$$

The leading term of the expectation $\langle X_t\rangle$ is the stochastic Stokes drift, which is $O\left(\varepsilon^2\right)$ [9]. Eq. (9) thus implies that $\langle f(X_t, s)\rangle$ must be $O(\varepsilon)$, and hence the second term in (8) is $O\left(\varepsilon^3\right)$. We will show later that the first term of (8), $\langle X_t f(X_t, t)\rangle$, is $O(\varepsilon)$, and therefore to the leading order for large t,

$$\operatorname{cov}(X_t, f(X_t, t)) \sim \langle X_t f(X_t, t)\rangle. \tag{10}$$

Using a Taylor series for $f(\cdot, t)$ as

$$f(X_t, t) = f(X_t^{(0)}, t) + \varepsilon X_t^{(1)} f'(X_t^{(0)}, t) + \varepsilon^2\left[\frac{(X_t^{(1)})^2}{2} f''(X_t^{(0)}, t) + X_t^{(2)} f'(X_t^{(0)}, t)\right] + O\left(\varepsilon^3\right) \tag{11}$$

where a dash on f denotes differentiation with respect to its first (position) argument, one obtains

$$\begin{aligned} X_t f(X_t, t) = {} & X_t^{(0)} f(X_t^{(0)}, t) + \varepsilon[X_t^{(1)} f(X_t^{(0)}, t) + X_t^{(0)} X_t^{(1)} f'(X_t^{(0)}, t)] \\ & + \varepsilon^2\left[X_t^{(2)} f(X_t^{(0)}, t) + (X_t^{(1)})^2 f'(X_t^{(0)}, t)\right. \\ & \left. + \frac{X_t^{(0)}(X_t^{(1)})^2}{2} f''(X_t^{(0)}, t) + X_t^{(0)} X_t^{(2)} f'(X_t^{(0)}, t)\right] + O\left(\varepsilon^3\right). \end{aligned} \tag{12}$$

Thus, taking expectation on both sides,

$$\begin{aligned} \langle X_t f(X_t, t)\rangle = {} & \langle X_t^{(0)} f(X_t^{(0)}, t)\rangle + \varepsilon[\langle X_t^{(1)} f(X_t^{(0)}, t)\rangle \\ & + \langle X_t^{(0)} X_t^{(1)} f'(X_t^{(0)}, t)\rangle] + O\left(\varepsilon^2\right). \end{aligned} \tag{13}$$

The above equation describes the diffusion behavior in any deterministic oscillatory flow field with the small ε. In the following, we shall examine specifically the condition whereby the flow field is induced by a monochromatic progressive wave.

3. Monochromatic progressive wave

We now focus on the basic case where the Eulerian flow field is a monochromatic progressive wave, i.e. $f(Y, t) = \cos(Y - t)$. The following standard results (see Appendix) are frequently used in deriving the expectations, namely, for a Gaussian random variable Y with zero mean and variance σ^2,

$$\begin{aligned} \langle \sin(Y - t)\rangle &= -\exp\left(-\frac{1}{2}\sigma^2\right) \sin t; \\ \langle \cos(Y - t)\rangle &= \exp\left(-\frac{1}{2}\sigma^2\right) \cos t \end{aligned} \tag{14}$$

$$\langle Y \sin(Y - t)\rangle = \sigma^2 \exp\left(-\frac{1}{2}\sigma^2\right) \cos t \tag{15}$$

$$\langle Y \cos(Y - t)\rangle = \sigma^2 \exp\left(-\frac{1}{2}\sigma^2\right) \sin t. \tag{16}$$

With these results, we now proceed to evaluate the three expectations in (13). First, using (5) and (16),

$$\langle X_t^{(0)} f(X_t^{(0)}, t)\rangle = \sigma^2 t \exp\left(-\frac{1}{2}\sigma^2 t\right) \sin t \tag{17}$$

which vanishes as $t \to \infty$. Note that the decay is exponentially fast at a rate of $\sigma^2/2$, which is generally true for other terms that will be discussed later. The implication is that the large time behavior may be assumed to be dominant when t is greater than some multiples of σ^2. Next,

$$\begin{aligned} \langle X_t^{(1)} f(X_t^{(0)}, t)\rangle &= \int_0^t \langle \cos(W_s - s) \cos(W_t - t)\rangle \,\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \langle \cos[(W_t - W_s) - (t - s)]\rangle \\ &\quad + \langle \cos[(W_t + W_s) - (t + s)]\rangle \,\mathrm{d}s \end{aligned} \tag{18}$$

where compound angle formula has been used. Note that $W_t = W_s + \Delta_{t-s}$, where Δ_{t-s} is the increment from s to t and is Gaussian with zero mean and variance $\sigma^2(t - s)$. Thus $W_t + W_s = 2W_s + \Delta_{t-s}$ is Gaussian with zero mean, and with a variance of