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A global compactness result for an elliptic equation with double singular terms

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ABSTRACT

In this paper, we establish a global compactness result for (P.S.) sequences of the variational functional of the elliptic problem

$$\begin{cases} -\Delta u - \dfrac{\mu}{|x|^2} u = \dfrac{1}{|x|^s} |u|^{2_s^*-2} u + \lambda u, & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded smooth domain with $0 \in \Omega$, $\mu \in [0, (n-2)^2/4)$, $s \in [0, 2)$ and $\lambda \in \mathbb{R}$ are constants. This extends the global compactness result of Cao and Peng (2003) to the case of elliptic problems with double singular critical terms. Our arguments adapt some refined Sobolev inequalities systematically developed quite recently by Palatucci and Pisante (2014) and blow-up analysis. In this way, our arguments turn out to be quite transparent and easy to be applied to many other problems.

1. Introduction and main result

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded smooth domain with $0 \in \Omega$, and $\mu \in [0, \bar{\mu})$, $\bar{\mu} = (n-2)^2/4$. In this paper, we consider the problem

$$\begin{cases} -\Delta u - \dfrac{\mu}{|x|^2} u = \dfrac{1}{|x|^s} |u|^{2_s^*-2} u + \lambda u, & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $0 \leq s < 2$ and $2_s^* = 2(n-s)/(n-2)$. When $s = 0$, we simply write 2^* in place of 2_0^*.

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Problems of type (1.1) have been extensively studied in the literature. When $\mu = s = 0$, problem (1.1) is reduced to

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In the seminal paper [1], Brezis and Nirenberg established the existence of a positive solution by showing that the corresponding variational functional $E_{BN}(u) = \frac{1}{2}\int_\Omega(|\nabla u|^2 - \lambda u^2) - \frac{1}{2^*}\int_\Omega |u|^{2^*}$ satisfies a local Palais–Smale (P.S. for short) condition. Their result is sharp in the sense that they found the critical value beyond which E_{BN} does not satisfy (P.S.) condition anymore. As this problem is typical, a complete description of global compactness condition for (P.S.) sequences of the functional E_{BN} turns out to be important. Finally, such a result was established by Struwe in his seminal paper [2].

In the case $s = 0$ and $0 < \mu < \bar{\mu}$, problem (1.1) is reduced to

$$\begin{cases} -\Delta u - \dfrac{\mu}{|x|^2}u = |u|^{2^*-2}u + \lambda u, & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The second term $\frac{\mu}{|x|^2}u$ is referred to as Hardy term in the literature, as it is closely related to the Hardy inequality

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \le \frac{1}{\bar{\mu}} \int_{\mathbb{R}^n} |\nabla u|^2, \qquad \forall\, u \in C_0^\infty(\mathbb{R}^n). \tag{1.3}$$

This term brings several mathematical difficulties. For instance, weak solutions of problem (1.2) turn out to be unbounded in any neighborhood of the origin, see e.g. [3]. For a complete description of asymptotic behaviors of weak solutions to elliptic equations with Hardy term, see e.g. Xiang [4]. The Hardy term also brings additional difficulty for blow-up analysis of (P.S.) sequences. In [5], Jannelli established existence of solutions to problem (1.2) for μ belonging to different sub-intervals of $(0, \bar{\mu})$. A crucial step of his arguments is to prove that the corresponding energy functional of problem (1.2) satisfies a local (P.S.) condition. For more existence results, we refer to e.g. [6,7]. Later, Cao and Peng [8] established a global compactness for problem (1.2). More global compactness results for various problems can be found, for instance, see Yan [9] for p-Laplacian critical problems, Palatucci and Pisante [10,11] for critical fractional Laplacian equations. For further applications of global compactness, we refer to e.g. [3,12] and the references therein.

In this paper, our aim is to extend the global compactness result of Cao and Peng [8] to problem (1.1) with $0 < s < 2$. It is well known that problem (1.1) is critical in the sense that the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2_s^*}(\Omega, |x|^{-s}\mathrm{d}x)$ is critical and noncompact. Problem (1.1) is variational. The corresponding variational functional is given by

$$E_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \frac{\mu}{|x|^2}u^2 - \lambda u^2\right) - \frac{1}{2_s^*}\int_\Omega \frac{1}{|x|^s}|u|^{2_s^*}, \qquad u \in H_0^1(\Omega).$$

It is standard to verify that E_λ satisfies a local but not global (P.S.) condition. To state our main result, let us introduce the limiting problem

$$-\Delta V - \frac{\lambda}{|x|^2}V = \frac{1}{|x|^s}|V|^{2_s^*-2}V, V \in D^{1,2}(\mathbb{R}^n), \tag{1.4}$$

and the corresponding energy functional

$$F(V) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla V|^2 - \frac{\mu}{|x|^2}V^2\right) - \frac{1}{2_s^*}\int_{\mathbb{R}^n} \frac{1}{|x|^s}|V|^{2_s^*} \qquad \text{for } V \in D^{1,2}(\mathbb{R}^n).$$

Here, $D^{1,2}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support, under the (semi-)norm '$\|\varphi\|_{D^{1,2}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla\varphi|^2\right)^{1/2}$. Our main result reads as follows.