Applied Mathematics Letters 87 (2019) 141–146

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

A priori estimates for the general dynamic Euler–Bernoulli beam equation: Supported and cantilever beams

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A R T I C L E   I N F O

Article history:
Received 28 June 2018
Received in revised form 30 July 2018
Accepted 30 July 2018
Available online 4 August 2018

Keywords:
Euler–Bernoulli beam equation
Weak and regular weak solution
A priori estimates
Uniqueness

A B S T R A C T

This work is a further development of weak solution theory for the general Euler–Bernoulli beam equation $\rho(x)u_{tt} + \mu(x)u_t + (r(x)u_{xx})_{xx} - (T_r(x)u_x)_x = F(x,t)$ defined in the finite dimension domain $\Omega_T := (0,l) \times (0,T) \subset \mathbb{R}^2$, based on the energy method. Here $r(x) = EI(x)$, $E > 0$ is the elasticity modulus and $I(x) > 0$ is the moment of inertia of the cross-section, $\rho(x) > 0$ is the mass density of the beam, $\mu(x) > 0$ is the damping coefficient and $T_r(x) \geq 0$ is the traction force along the beam. Two benchmark initial boundary value problems with mixed boundary conditions, corresponding to supported and cantilever beams, are analyzed. For the weak and regular weak solutions of these problems a priori estimates are derived under the minimal conditions. These estimates in particular imply the uniqueness of the solutions of both problems.

1. Introduction

The Euler–Bernoulli beam equation is used to model bending vibration of many mechanical systems from industry and engineering [1]. The need to control the dynamics of these systems has made analysis and simulation of such systems an important research area. Vibration problems related to the static and dynamic response of beams have been studied since the end of the 18th century, beginning with the work of Stokes [2] and Aitken [3].

The *first mathematical model* corresponds to the following initial boundary value problem for a simply supported dynamic Euler–Bernoulli beam under the moving load dynamic $F(x,t)$:

$$\begin{cases} \rho(x)u_{tt} + \mu(x)u_t + (r(x)u_{xx})_{xx} - (T_r(x)u_x)_x = F(x,t), \ (x,t) \in (0,l) \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in (0,l), \\ u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0, \ t \in (0,T). \end{cases} \tag{1}$$

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https://doi.org/10.1016/j.aml.2018.07.038
0893-9659/

The *second mathematical model* corresponds to the following initial boundary value problem for a cantilever Euler–Bernoulli beam under the transfer shear force $g(t)$ and a constant T_r $(0 \le T_r < \infty)$:

$$\begin{cases} \rho(x)u_{tt} + \mu(x)u_t + (r(x)u_{xx})_{xx} - T_r u_{xx} = 0,\ (x,t) \in (0,l)\times(0,T), \\ u(x,0) = u_t(x,0) = 0,\ x \in (0,l), \\ u(0,t) = u_x(0,t) = u_{xx}(l,t) = 0,\ -(r(x)u_{xx})_x|_{x=l} = g(t),\ t \in (0,T). \end{cases} \tag{2}$$

Although some a priori estimates for problem (1) are discussed in [4–6], to our knowledge, no systematic research exists addressing the systematic study of weak and regular weak solutions of problems (1) and (2). Due to the limited scope of this paper, here only these problems are analyzed. We hope that our paper can motivate further study in this direction.

The paper is organized as follows. A priori estimates for problems (1) and (2) are discussed in Sections 2 and 3, respectively. Some concluding remarks are given in Section 4.

2. Estimates for the weak and regular weak solutions of (1)

We assume that functions $\rho(x)$, $r(x)$, $\mu(x)$, $T_r(x)$, $u_0(x)$, $u_1(x)$ and $g(t)$ satisfy the following conditions:

$$\begin{cases} \rho(x), r(x), \mu(x), T_r(x) \in L^\infty(0,l),\ g(t) \in H^1(0,T),\ g(0) = 0, \\ u_0(x) \in H^2(0,l), u_1(x) \in L^2(0,l),\ F(x,t) \in L^2(0,T;L^2(0,l)), \\ 0 < \rho_0 \le \rho(x) \le \rho_1,\ 0 < r_0 \le r(x) \le r_1, \\ 0 < \mu_0 \le \mu(x) \le \mu_1,\ 0 \le T_{r_0} \le T_r(x) \le T_{r_1}. \end{cases} \tag{3}$$

It can be proved that under these conditions there exists a unique weak solution of problem (1) defined as $u \in L^2(0,T;\mathcal{V}(0,l))$, $u_t \in L^2(0,T;L^2(0,l))$, $u_{tt} \in L^2(0,T;H^{-2}(0,l))$, where $\mathcal{V}(0,l) := \{v \in H^2(0,l) :\ v(0) = v(l) = 0\}$ for problem (1) and $\mathcal{V}(0,l) := \{v \in H^2(0,l) :\ v(0) = v'(0) = 0\}$ for problem (2), respectively. This weak solution also belongs to $C([0,T];H^1(0,l))$, as it follows from Theorem 4, Ch. 5 of [6].

Theorem 1. *Let conditions (3) hold. Then for the weak solution of problem (1) the following estimates hold:*

$$\|u_t\|^2_{L^2(0,T;\,L^2(0,l))} \le \left[\|F\|^2_{L^2(0,T;L^2(0,l))} + 2E_0\right]\left[\exp(T/\rho_0) - 1\right], \tag{4}$$

$$\|u_{xx}\|_{L^\infty(0,T;\,L^2(0,l))} \le r_0^{-1}\left[\|F\|^2_{L^2(0,T;L^2(0,l))} + 2E_0\right]\exp(T/\rho_0), \tag{5}$$

where

$$E_0 = \frac{1}{2}\int_0^l \left[\rho(x)u_1^2(x) + r(x)(u_0''(x))^2 + T_r(x)(u_0'(x))^2\right]dx. \tag{6}$$

Proof. Multiply both sides of Eq. (1) by $u_t(x,t)$, use the identity

$$(r(x)u_{xx})_{xx}u_t \equiv [(r(x)u_{xx})_x u_t - r(x)u_{xx}u_{xt}]_x + \frac{1}{2}r(x)\left(u_{xx}^2\right)_t, \tag{7}$$

integrate over $\Omega_t := (0,l)\times(0,t)$ and apply the integration by parts formula. Taking into account the initial and boundary conditions in (1) we obtain the following *energy identity*:

$$\frac{1}{2}\int_0^l \left[\rho(x)u_t^2 + r(x)u_{xx}^2 + T_r(x)u_x^2\right]dx + \int_0^t\int_0^l \mu(x)u_\tau^2 dx d\tau = \int_0^t\int_0^l F(x,\tau)u_\tau dx d\tau + E_0, \tag{8}$$

for a.e. $t \in [0,T]$, where $E_0 > 0$ is defined by (6). Using the inequality $2ab \le a^2+b^2$ in the first right-hand-side integral of (8) we deduce that

$$\int_0^l u_t^2(x,t)dx \le \frac{1}{\rho_0}\int_0^t\int_0^l u_\tau^2(x,\tau)dx d\tau + \frac{1}{\rho_0}\int_0^t\int_0^l F^2(x,\tau)dx d\tau + \frac{2}{\rho_0}E_0,\ t \in [0,T].$$