Applied Mathematics Letters 86 (2018) 22–29

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Dual virtual element method in presence of an inclusion

Alessio Fumagalli

University of Bergen, Department of Mathematics, Allegaten 41, 5007 Bergen, Norway

ARTICLE INFO

Article history:
Received 11 April 2018
Received in revised form 5 June 2018
Accepted 5 June 2018
Available online 18 June 2018

Keywords:
VEM
Darcy flow
Fractured porous media
Inclusion

ABSTRACT

We consider a Darcy problem for saturated porous media written in dual formulation in presence of a fully immersed inclusion. The lowest order virtual element method is employed to derive the discrete approximation. In the present work we study the effect of cells with cuts on the numerical solution, able to geometrically handle in a more natural way the inclusion tips. The numerical results show the validity of the proposed approach.

1. Introduction

Single-phase flow in fractured porous media is a challenging problem involving different aspects, *i.e.* the derivation of proper mathematical models to describe fracture and surrounding porous media flow and the subsequent discretization with ad-hoc numerical schemes. One of the most common approaches is to consider fractures as co-dimensional objects and derive proper reduced models and coupling conditions to describe the flow in the new setting. A hybrid dimensional description of the problem is thus introduced, see [1–5]. In presence of multiple fractures, forming a complex network, grid creation may become challenging and the number of cells or their shape may not be satisfactory for complex application, *e.g.*, the benchmark study proposed in [6].

In the present work we simplify the problem considering a single fracture and substituting with an inclusion, *i.e.* the flow problem in the fracture is not considered but its effect of flow in its normal direction. The inclusion is an internal condition for the problem. We consider two extreme cases: perfectly permeable fracture (infinite normal permeability) and impermeable fracture (zero normal permeability). The pressure imposed on both sides of the inclusion determines these two cases.

The virtual element method (VEM), introduced in [7–15], is able to discretize the problem on grids with rather general cell shape. The theory developed in the aforementioned works considers star shaped cells. In the present study, the lowest order VEM is considered in presence of cells with an internal cut for the approximation at the tips of the inclusion.

E-mail address: alessio.fumagalli@uib.no.

https://doi.org/10.1016/j.aml.2018.06.004

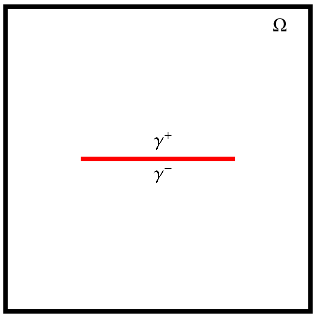

Fig. 1. Domain.

The paper is organized as follows. In Section 2 the mathematical model and its weak formulation are presented. Section 3 introduces the discrete formulation of the problem. Numerical examples are reported in Section 4 for both the extreme cases. The work finishes with conclusions in Section 5.

2. Mathematical model

Let us set $\Omega \subset \mathbb{R}^2$ a regular domain representing a porous media. We consider the Darcy model for single phase flow in a saturated porous media written in dual formulation, namely

$$\boldsymbol{u} + K\nabla p = \boldsymbol{0} \text{ in } \Omega \quad \wedge \quad \nabla \cdot \boldsymbol{u} = f \text{ in } \Omega \quad \wedge \quad p = 0 \text{ on } \partial\Omega. \tag{1a}$$

The unknowns are: $\boldsymbol{u}$ the Darcy velocity and p the fluid pressure. In (1a) K represents the permeability matrix, symmetric and positive defined, and f a scalar source or sink term. To keep the presentation simple we consider only homogeneous boundary condition for the pressure at the outer boundary of Ω, denoted by $\partial\Omega$. Coupled to (1a) we are interested to model an immersed inclusion γ, see Fig. 1 as an example. With an abuse of notation, $\partial\Omega$ does not include γ which represents an internal boundary for Ω. We assume $\partial\Omega \cap \gamma = \emptyset$. It is possible to define a unique normal $\boldsymbol{n}$ associated to γ and two different sides of γ with respect to the direction of $\boldsymbol{n}$. We indicate them as γ^+ and γ^-, with $\boldsymbol{n}^+ = \boldsymbol{n}$ and $\boldsymbol{n}^- = -\boldsymbol{n}$ the associated normals. It is important to note that geometrically γ, γ^+, and γ^- are indeed the same objects but introducing the two sides helps us to impose different internal conditions. We have

$$p = p^+ \text{ on } \gamma^+ \quad \wedge \quad p = p^- \text{ on } \gamma^-. \tag{1b}$$

We define the Hilbert spaces $Q = L^2(\Omega)$ and $V = H_{\text{div}}(\Omega)$. By standard arguments the weak formulation of (1) reads find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = J(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V \quad \wedge \quad b(\boldsymbol{u}, q) = F(q) \quad \forall q \in Q. \tag{2}$$

The bilinear forms and functionals are defined as

$$a(\cdot,\cdot) : V \times V \to \mathbb{R} \quad \text{s.t.} \quad a(\boldsymbol{w}, \boldsymbol{v}) := (K^{-1}\boldsymbol{w}, \boldsymbol{v})_\Omega, \qquad b(\cdot,\cdot) : V \times Q \to \mathbb{R} \quad \text{s.t.} \quad b(\boldsymbol{v}, q) := -(\nabla \cdot \boldsymbol{v}, q)_\Omega,$$
$$J(\cdot) : V \to \mathbb{R} \quad \text{s.t.} \quad J(\boldsymbol{v}) := -\langle \boldsymbol{v}^+ \cdot \boldsymbol{n}^+, p^+ \rangle_{\gamma^+} - \langle \boldsymbol{v}^- \cdot \boldsymbol{n}^-, p^- \rangle_{\gamma^-}, \quad F(\cdot) : Q \to \mathbb{R} \quad \text{s.t.} \quad F(q) := -(f, q)_\Omega$$

where we have indicated by $(\cdot,\cdot)_\Omega$: $Q \times Q \to \mathbb{R}$ the scalar product in Q and the duality pairings are defined as $\langle \cdot,\cdot \rangle_{\gamma^\pm} : H^{-\frac{1}{2}}(\gamma^\pm) \times H^{\frac{1}{2}}(\gamma^\pm) \to \mathbb{R}$. We are assuming $p^\pm \in H^{\frac{1}{2}}(\gamma^\pm)$ and $f \in L^2(\Omega)$. Following [14] problem (2) is well posed.