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Efficient computation of matrix power-vector products: application for space-fractional diffusion problems

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Abstract

A novel algorithm is proposed for computing matrix-vector products $A^\alpha \mathbf{v}$, where A is a symmetric positive semidefinite sparse matrix and $\alpha > 0$. The method can be applied for the efficient implementation of the matrix transformation method to solve space-fractional diffusion problems. The performance of the new algorithm is studied in a comparison with the conventional MATLAB subroutines to compute matrix powers.

Keywords: matrix powers, matrix transformation method, binomial series, space-fractional diffusion
2010 MSC: 65N12, 65N15, 65N30

1. Introduction

In the last two decades, the numerical simulation of (space-) fractional diffusion became an important topic in the numerical PDE's starting with the poineering paper [1]. Fractional dynamics was observed in a wide range of life sciences [2], earth sciences [3] and financial processes [4]. A frequently used and meaningful mathematical model [5] of these phenomena is offered by the fractional Laplacian operator [6]. For solving these problems numerically, the so-called matrix transformation method was proposed in [7] and [8]. Accordingly, a mathematical analysis was presented to prove the convergence of this simple approach in case of finite difference [9] and finite element discretization.

The bottleneck of the matrix transformation approach is the computation of the fractional power of the matrices arising from the finite element or the finite difference discretization of the negative Laplacian operators. This is why many authors use a more technical approach [6].

Several algorithms were proposed to compute this matrix power. We refer to the latest approach in [10] and its reference list on the earlier developments. Due to its importance, MATLAB has also a built-in subroutine `mpower.m`.

To bypass this costly procedure, the aim of the present article is to develop a simple and fast numerical method to compute matrix-vector products of type $A^\alpha \mathbf{v}$. This idea was also used in [11] by developing the frequently used Matlab subroutine `expmv.m` to compute matrix exponential-vector products.

1.1. Mathematical preliminaries

The main motivation of our study is the efficient numerical solution of the initial-boundary value problem

$$\begin{cases} \partial_t u(t,\mathbf{x}) = -(-\Delta)^\alpha u(t,\mathbf{x}) & t \in (0,T),\ \mathbf{x} \in \Omega \\ u(0,\mathbf{x}) = u_0(\mathbf{x}) & t \in (0,T),\ \mathbf{x} \in \Omega, \end{cases} \tag{1}$$

where $\Delta = \Delta_{\mathcal{D}}$ or $\Delta = \Delta_{\mathcal{N}}$ denotes the Dirichlet or the Neumann Laplacian on the bounded Lipschitz domain Ω and $u_0 \in L_2(\Omega)$ is given. Note that $-\Delta_{\mathcal{D}}^{-1} : L_2(\Omega) \to L_2(\Omega)$ and $-\Delta_{\mathcal{N}}^{-1} : L_2(\Omega)/\mathbb{R} \to L_2(\Omega)$ are compact and self-adjoint, so that its fractional power makes sense.

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