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The cauchy problem for linear inhomogeneous wave equations with variable coefficients

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Abstract

In this paper, we present a new analytical formula for the Cauchy problem of the linear inhomogeneous wave equation with variable coefficients. The formula gives a much simpler solution than that given by the classical Poisson formula. The derivation is based on Duhamel's Principle and the theory of pseudodifferential operator. An example is solved by using the formula to illustrate the feasibility.

Keywords: Cauchy problem, inhomogeneous wave equation, variable coefficient

2010 Mathematics Subject Classification. 35L15, 35S10, 65L05, 65L06

1. Introduction

The history of partial differential equations can date back to the 18th century when the first one-dimensional wave equation $u_{tt} = u_{xx}$ was introduced by d'Alembert as a model of a vibrating string [1]. Giving its initial displacement $u(x,0) = \varphi(x)$ and velocity $u_t(x,0) = \psi(x)$, it is known that an explicit solution to the Cauchy, or initial value, problem of the wave equation is

$$u(x,t) = \frac{1}{2}(\varphi(x-t) + \varphi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)d\xi.$$

If an external force $f(x,t)$ (per unit mass) applied on the string, then the equation becomes to an inhomogeneous wave equation $u_{tt} = u_{xx} + f(x,t)$. In this case, the solution which can be derived by using Duhamel's Principle is

$$u(x,t) = \frac{1}{2}(\varphi(x-t) + \varphi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f(\sigma,\tau)d\sigma d\tau.$$

Various methods have been utilized to solve the Cauchy problem for higher dimensional wave equation such as the Hadamard method of descent for $n = 2$, the method of spherical means for $n = 3$ [2], the analytic continuation of an integral of fractional order [3] and the finite part of a divergent integral [4], etc. A formal series solution has been derived by using a classical power series in [5] and the same formula is obtained in [6] by using the Adomian decomposition method. A direct derivation of the spherical means solution for arbitrary dimension n is presented using Fourier transform in [7].

Consider the Cauchy problem for the following linear inhomogeneous wave equation with variable coefficients in $\mathbb{R}^n$

$$\begin{cases} u_{tt} + Lu = f(x,t), & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

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