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Dissipativity of semilinear time fractional subdiffusion equations and numerical approximations

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The dissipativity with a bounded absorbing set in $L^2(\Omega)$ for time-fractional nonlinear sub-diffusion equation is investigated. A numerical scheme based on the $L1$ method for time fractional derivative and the standard finite element method in space direction is presented. The proposed numerical scheme can preserve the dissipativity as the continuous equation. A numerical example is included to show the asymptotic behavior of the numerical method.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Dirichlet boundary values $\partial\Omega$ and consider the following time fractional sub-diffusion equation

$$D_t^\alpha u - d\Delta u + f(u) = 0, \qquad x \in \Omega, \alpha \in (0,1), \tag{1}$$

subject to the initial/boundary value conditions: $u(x,0) = u_0$ for $x \in \Omega$ and $u(x,t) = 0$ for $x \in \partial\Omega$. Here the symbol D_t^α denotes the Caputo time fractional derivative: $D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u^{(1)}(\tau)}{(t-\tau)^\alpha} d\tau$ for $t > 0$. Here we take the nonlinear term $f(u)$ to be a polynomial of odd degree with a positive leading coefficient given by

$$f(u) = \sum_{j=0}^{2p-1} b_j u^j, \quad b_{2p-1} > 0, \quad p \in \mathbb{N}^+. \tag{2}$$

Polynomial nonlinearity is a very common and typical case, including many important physical models such as FitzHugh–Nagumo equations. Other examples can be found in [1–3].

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Fractional calculus originate in the seventeenth century and have the same long history as the integer-order calculus. Fractional differential equations have achieved all-round development since the middle of last century and are now widely applied in many fields such as the physics, finance and biology [4,5]. The most successful application is the fractional partial differential equations (PDEs) in various anomalous diffusion models, which describe a diffusion process where the mean square displacement of a particle grows slower or faster than that in the normal diffusion process. The anomalous diffusion has been proved in many experiments, which show the superiority of fractional differential equations in the characterization of complex processes. The representation formulas of fundamental solutions and decay properties of linear time fractional subdiffusion equations have been well studied such as in [6–8]. The regularity of weak solutions to such problems was recently considered in [9].

When $\alpha = 1$ Eq. (1) recovers the classical parabolic equation. The natural object to describe the long time behavior of solutions to nonlinear evolutionary PDEs is the global attractor. There are plenty of achievements about the exponential attractors for integer-order reaction–diffusion equations. For more details, please refer to the monographs and lectures in [1–3,10–12] and references therein. In the unbounded domain, Babin and Nicolaenko [9] proved the existence of exponential attractors for reaction–diffusion systems and estimated their fractal dimension. In the bounded domain, Efendiev and Miranville [10] obtained finite dimensions of a global attractor in $L^\infty(\Omega)$ with nonlinearity f satisfying the arbitrary growth with inhomogeneous term. In [2], Zhong, Yang and Sun introduced a new concept called the norm-to-weak continuous semigroup in a Banach space and obtained the existence of the global attractor for a nonlinear reaction–diffusion equation with the polynomial growth nonlinearity of arbitrary order, and that the global attractors are obtained in $L^p(\Omega), H_0^1(\Omega)$ and $H^2(\Omega) \cap H_0^1(\Omega)$, respectively. In [3], Zhong and Zhong studied the exponential attractors for the equation with arbitrary polynomial growth nonlinearity and inhomogeneous term and obtained the exponential attractor in $L^2(\Omega), H_0^1(\Omega), L^{2p-2}(\Omega)$, respectively.

As far as we know, there are few studies on the dynamical property of long time behavior for fractional PDEs in the literature due to the difficulties caused from the nonlocal nature of fractional derivatives. The main purpose of this paper is to establish the dissipativity of a class of nonlinear time fractional PDEs and to develop numerical methods which can inherit this property. The main feature of dissipative systems is the presence of mechanisms of energy dissipation, which can lead to quite complicated limit regimes and structures [11]. The models of dissipative differential equations often arise in the fields of physics and engineering; refer e.g., to [1,11] for details. The dissipativity and contractivity of fractional ordinary differential equations and fractional functional differential equations were recently developed in [13,14]. The research on the long time behavior for time fractional differential equations is further studied in this article.

Generally speaking, the long time decay rate of classical integer-order differential equations is exponential while fractional differential equations is often polynomial. This is an important distinction between the integer-order differential equations and the fractional differential equations. This fact can be primarily seen from the solution of the classical linear diffusion equation and the corresponding fractional version. Consider the one dimension classical diffusion equation $u_t = u_{xx}$ for $t > 0$ and $x \in \Omega$, and subject to the initial value $u(x,0) = u_0$ and homogeneous boundary condition $u(x,t) = 0, x \in \partial\Omega$. Then the solution is $u(x,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\langle \phi_j, u_0\rangle \phi_j$, where $\{\phi_j\}_{j=0}^{\infty}$ and $\{\lambda_j\}_{j=0}^{\infty}$ are the eigenfunctions and eigenvalues of Laplace operator Δ. This implies the well-known long time exponential decay rate of the solution $u(x,t)$. But for the fractional counterpart $\partial_t^\alpha u = u_{xx}$ with the same initial/boundary conditions, the solution becomes $u(x,t) = \sum_{j=0}^{\infty} E_{\alpha,1}(-\lambda_j t^\alpha)\langle \phi_j, u_0\rangle \phi_j$, where $E_{\alpha,1}$ is the generalized Mittag-Leffler function with the property $|E_{\alpha,\beta}(z)| \le \frac{C}{1+|z|}$, which indicates that $\|u(x,t)\|^2_{L^2(\Omega)} = \sum_{j=0}^{\infty} \mid (\phi_j, u_0) E_{\alpha,1}(-\lambda_j t^\alpha)|^2 \le \frac{C_1\|u_0\|^2_{L^2(\Omega)}}{t^{2\alpha}}$. It means that the solution of fractional subdiffusion equation has the long time polynomial decay rate, i.e., $u(x,t) = O(t^{-\alpha})$ as $t \to +\infty$. The specific details can be found in [15]. Thus, the fractional subdiffusion equation often has much slower speed than classical diffusion and has a long tail.