Applied Mathematics Letters 86 (2018) 284–290

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

A simple fully well-balanced and entropy preserving scheme for the shallow-water equations

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ARTICLE INFO

Article history:
Received 30 April 2018
Received in revised form 9 July 2018
Accepted 9 July 2018
Available online 25 July 2018

MSC:
65M08
65M12

Keywords:
Shallow-water equations
Godunov-type schemes
Well-balanced schemes
Moving steady states
Entropy-satisfying schemes

ABSTRACT

In this communication, we consider a numerical scheme for the shallow-water system. The scheme under consideration has been proven to preserve the positivity of the water height and to be fully well-balanced, i.e. to exactly preserve the smooth moving steady state solutions of the shallow-water equations with the topography source term. The goal of this work is to prove a discrete entropy inequality satisfied by this scheme.

1. Introduction

In this work, we consider the shallow-water approximation of free-surface flows in a longitudinal channel. In one space dimension, this model is governed by the following system:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z, \end{cases} \tag{1}$$

where $h(x,t)$ is the water height, $q(x,t)$ is its discharge (equal to hu, where u is the water velocity), g is the gravity constant and Z is the smooth given bottom topography. In order to shorten the notations, we rewrite (1) under the classical form of a conservation law with a source term $\partial_t W + \partial_x F(W) = S(W)$, where

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https://doi.org/10.1016/j.aml.2018.07.013
0893-9659/

we have set:

$$W = \begin{pmatrix} h \\ q \end{pmatrix}, \quad F(W) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + \dfrac{1}{2}gh^2 \end{pmatrix}, \quad S(W) = \begin{pmatrix} 0 \\ -gh\partial_x Z \end{pmatrix}.$$

In the present work, we assume that the flow is always far from dry areas. The conserved variables W thus lie in the set of admissible states Ω, which prescribes a physically admissible positive water height:

$$\Omega = \{W = {}^t(h, q) \in \mathbb{R}^2 \mid h > 0\}.$$

In addition, note that the following natural entropy inequality, satisfied by the admissible entropy weak solutions, arises from this system:

$$\partial_t \eta(W) + \partial_x G(W) \leq -gq\partial_x Z, \tag{2}$$

where the entropy η and the entropy flux G are defined by:

$$\eta(W) = \frac{1}{2}\frac{q^2}{h} + \frac{1}{2}gh^2 \text{ and } G(W) = \frac{q}{h}\left(\frac{1}{2}\frac{q^2}{h} + gh^2\right).$$

Note that, according to [1], the non-conservative entropy inequality (2) can be recast under the following equivalent conservative form:

$$\partial_t \left(\eta(W) + ghZ\right) + \partial_x \left(G(W) + gqZ\right) \leq 0. \tag{3}$$

Preserving the steady state solutions of the shallow-water equations, i.e. solutions of (1) such that $\partial_t W = 0$, has been a major challenge of the last two decades. The steady states at rest, describing motionless water over a possibly complex bottom topography, are obtained by assuming $q = 0$ as well as $\partial_t W = 0$, to get $\partial_x(h + Z) = 0$. These steady states have been the focus of much work, and several relevant numerical schemes have been developed (see for instance [2–4], but this list is far from being exhaustive). Such steady solutions are widely encountered in real-world applications, and being able to preserve them is crucial for a numerical method. Conversely, less work has been undertaken on so-called fully well-balanced schemes, which exactly preserve the smooth moving steady state solutions, defined by:

$$\begin{cases} q = \text{cst}, \\ \dfrac{q^2}{2h^2} + g\left(h + Z\right) = \text{cst}. \end{cases} \tag{4}$$

In particular, few first-order schemes have been developed (see for instance [5–7]). Nice properties for a scheme to possess are, in addition to being well-balanced, the preservation of the admissible set Ω (i.e. the preservation of the water height positivity, also called the robustness property) and a discrete analogue to the entropy inequality (2).

In [6], the authors derive a scheme with the three previous properties. However, in practice, this scheme is computationally too costly, since it involves finding the roots of a fifth-order polynomial. The scheme proposed in [7] corrects this cost shortcoming by introducing an approach taking into account a generic source term, leading to a suitable linearization. However, an entropy inequality was not exhibited. The goal of the present work is to establish an entropy inequality satisfied, in some sense to be prescribed, by the numerical scheme proposed in [7].

2. Presentation of the numerical scheme

In this Section, for the sake of completeness, we give the numerical scheme developed in [7]. It falls within the framework of finite volume schemes, and more specifically of Godunov-type schemes (see [8] for