On the relaxation of the Maxwell–Stefan system to linear diffusion[☆]

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A R T I C L E   I N F O

A B S T R A C T

In this note, we rigorously prove the relaxation limit of the Maxwell–Stefan system to a system of heat equations when all binary diffusion coefficients tend to the same positive value.

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1. Introduction

The Maxwell–Stefan equations have been written in the nineteenth century [1,2], but the interest in their rigorous mathematical study is very recent and not yet complete. After some works, mainly devoted to the matrix formulation of the gradient–flux relationships and described in [3], the study of existence and uniqueness has been carried out in [4–9], the formal derivation of multicomponent diffusion equations from the Boltzmann system has been investigated in [10–15] and some numerical discretizations of the Maxwell–Stefan system have been proposed in [5,16].

Despite the advances of the last years, the aforementioned results are far to be complete and many questions are still waiting for a satisfactory answer. Among them, we cite the rigorous study of the relaxation of the Maxwell–Stefan system, under the condition of equimolar diffusion, to a system of heat equations when all binary diffusion coefficients tend to the same positive constant value. In this note, we intend to give a contribution on this last question and fill in the gap in the literature.

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2. The problem and the main result

Let $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a bounded domain with regular boundary of class C^2 and $\varepsilon > 0$. Consider a collection of ε-dependent strictly positive and symmetric coefficients k_{ij}^ε (i.e. $k_{ij}^\varepsilon = k_{ji}^\varepsilon > 0$ for all $i, j = 1, \ldots, I \in \mathbb{N}$).

The Maxwell–Stefan equations describe a gaseous mixture composed of $I \geq 3$ interacting species. The cross-diffusion relationships link the ε-dependent unknown densities and fluxes $(c_i^\varepsilon, J_i^\varepsilon)$, $i = 1, \ldots, I \in \mathbb{N}$, between themselves:

$$\begin{cases} \partial_t c_i^\varepsilon + \nabla_x \cdot J_i^\varepsilon = 0, & (t, x) \in \mathbb{R}^+ \times \Omega \\ \nabla_x c_i^\varepsilon = -\sum_{j \neq i} k_{ij}^\varepsilon (c_j^\varepsilon J_i^\varepsilon - c_i^\varepsilon J_j^\varepsilon). \end{cases} \tag{2.1}$$

Moreover, the fluxes satisfy the equimolar diffusion relationship

$$\sum_{i=1}^I J_i^\varepsilon = 0. \tag{2.2}$$

The ε-dependent coefficients k_{ij}^ε represent the set of binary diffusion coefficients of the gaseous mixture whose behavior is described by the Maxwell–Stefan equations (2.1). Note that the diagonal coefficients k_{ii}^ε ($i = 1, \ldots, I$) play no role in the system.

System (2.1) is supplemented with appropriate initial and boundary conditions. In what follows, we suppose that the solution of (2.1) satisfies homogeneous Neumann boundary conditions, so that we have:

$$\begin{aligned} (c_1^\varepsilon(0, x), \ldots, c_I^\varepsilon(0, x)) &= (c_1^{\text{in}}(x), \ldots, c_I^{\text{in}}(x)) \in (L^\infty(\Omega))^I, \\ J_i^\varepsilon(t, x) \cdot n_x &= 0, \ (t, x) \in (0, \infty) \times \partial\Omega, \ i = 1, \ldots, I, \end{aligned} \tag{2.3}$$

where $n_x \in \mathbb{S}^{d-1}$ is the outward normal unit vector to the domain Ω starting from a given point $x \in \partial\Omega$. Since the Maxwell–Stefan system (2.1) is written in terms of molar fractions, we assume that all $c_i^{\text{in}} \geq 0$ and

$$\sum_{i=1}^I c_i^{\text{in}}(x) = 1.$$

Because of the symmetry of the binary diffusion coefficients, it is easy to see, from (2.1) and the equimolar diffusion condition (2.2), that

$$\sum_{i=1}^I c_i^\varepsilon(t, x) = 1 \tag{2.4}$$

for a.e. $(t, x) \in \mathbb{R}^+ \times \Omega$. When all the binary diffusion coefficients are equal (i.e. when $k_{ij}^\varepsilon = \kappa$ for all i, j), it is easy to see that the Maxwell–Stefan equations (2.1), together with Eqs. (2.2) and (2.4), are equivalent to a system of heat equations for $c_i^\varepsilon(t, x)$. Indeed, thanks to the equimolar diffusion condition, we have that

$$J_i^\varepsilon = -\sum_{j \neq i} J_j^\varepsilon$$

and hence, from (2.4) and the second relationship in (2.1), we deduce that

$$J_i^\varepsilon = -\frac{1}{\kappa} \nabla_x c_i^\varepsilon$$

for all $i = 1, \ldots, I$, which is the standard form of Fick's law [17,18]. It leads, thanks to the first equation in (2.1), to the system of heat equations

$$\partial_t c_i^\varepsilon = \frac{1}{\kappa} \Delta_x c_i^\varepsilon, \qquad i = 1, \ldots, I, \tag{2.5}$$