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Abstract

We introduce auxiliary points to obtain domains of global convergence for Newton's method without imposing convergence conditions on the local solution of equation to solve, as local convergence results do.

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1 Introduction

The solutions of nonlinear equations are rarely found in closed form, so that we usually use iterative methods to solve them. It is well known that one of the first iterative methods used to solve nonlinear equations was the method of successive approximations, which is based on the Fixed Point Theorem. So, approximating a solution x^* of equation $F(x) = 0$, where F is an operator defined on a nonempty open convex domain Ω of a Banach space X with values in a Banach space Y, is equivalent to approximating, for example, a fixed point of equation $x + F(x) = 0$. Remember that if $G : \mathcal{D} \to \mathcal{D}$, where $\mathcal{D}$ is a nonempty closed subset of X, is a contractive operator, then there exists a unique $y^* \in \mathcal{D}$ such that $G(y^*) = y^*$ and the method of successive approximations, $x_{n+1} = G(x_n)$, with $x_0 \in \mathcal{D}$, converges to the fixed point y^*. One of the main features of this fact is that we obtain the approximation of fixed point y^* from any point $x_0 \in \mathcal{D}$.

However, if we consider the most well-known iterative method, Newton's method,

$$x_{n+1} = N_F(x_n) = x_n - [F'(x_n)]^{-1}F(x_n), \quad n \geq 0, \quad \text{with } x_0 \text{ given},$$

where $F'(x_n)$ is the first Fréchet derivative of the operator $F(x)$ at the point x_n and $[F'(x_n)]^{-1}$ is the inverse operator, we cannot apply in general the Fixed Point Theorem to operator N_F to guarantee the convergence of Newton's method and one of the difficulties that involves the application of the method is then the location of an appropriate starting point x_0 from which it converges.

The study of convergence of Newton's method is usually done in two ways, by means of semilocal and local convergence studies. The semilocal convergence analysis is based on the information around initial point x_0 and gives criteria ensuring the convergence of Newton's method, while the local one is based on the information around solution x^* and gives estimates of the radius of local convergence ball [1].

The main difference between both types of analysis of convergence of Newton's method is that conditions on the starting point x_0 are imposed in results of semilocal convergence and the solution is then located in a ball centered at x_0, while conditions on the solution x^* are imposed in the results of local convergence and a convergence ball is then obtained that gives us an idea of the accessibility of the method to x^*. Then, the main difficulty of semilocal convergence results lies in the location of good starting points for Newton's method and that of local convergence results is the solution x^* of the equation is usually unknown.

The aim of this work is to obtain domains of global convergence for Newton's method similar to those obtained from the local convergence results, but without demanding conditions on the solution x^*, that is generally unknown. For this, we use auxiliary points to impose convergence conditions.

Throughout the paper, we denote $\overline{B(x,\varrho)} = \{y \in X; \|y - x\| \leq \varrho\}$ and $B(x,\varrho) = \{y \in X; \|y - x\| < \varrho\}$ and the set of bounded linear operators from Y to X by $\mathcal{L}(Y,X)$.