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Representation of solution of a Riemann–Liouville fractional differential equation with pure delay[☆]

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ABSTRACT

This paper derives a representation of a solution to the initial value problem for a linear fractional delay differential equation with Riemann–Liouville derivative. We apply the method of variation of constants to obtain the representation of a solution via a delayed Mittag-Leffler type matrix function.

1. Introduction

Recently, Khusainov and Shuklin [1] and Diblík and Khusainov [2,3] introduce a new concept, delayed exponential matrix function, which is used to seek a representation of a solution to a linear time-invariant continuous/discrete delay equation. For more contributions of the representation of solutions, the stability, and the control theory for linear time-invariant continuous/discrete delay systems, we refer to [4–9] and the references.

Inspired by [7,8], we seek a representation of a solution to a linear fractional delay differential equation with Riemann–Liouville derivative whose the initial condition involving a singular kernel that is different from the standard initial condition for a Caputo fractional delay differential equation.

In this paper, we study a fractional delay differential equation of the form:

$$\begin{cases} (\mathbb{D}^{\alpha}_{-\tau^+}y)(x) = By(x-\tau) + f(x),\ B \in \mathbb{R}^{n\times n},\ x \in (0,T],\ \tau > 0, \\ y(x) = \omega(x),\ \omega(x) \in \mathbb{R}^n\ \ -\tau \le x \le 0, \\ (\mathbb{I}^{1-\alpha}_{-\tau^+}y)(-\tau^+) = \omega(-\tau),\ \omega(-\tau) \in \mathbb{R}^n, \end{cases} \tag{1}$$

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where $\mathbb{D}^{\alpha}_{-\tau^+}y$ denotes the Riemann–Liouville derivative of order $\alpha \in (0,1)$ (see Definition 2.1), $\mathbb{I}^{1-\alpha}_{-\tau^+}y$ denotes the Riemann–Liouville fractional integral of order $1-\alpha$ (see Definition 2.2), $T = k^*\tau$ for a fixed $k^* \in \mathbb{N}^+ := \{1, 2, \ldots\}$, τ is a fixed delay time, $f \in C([-\tau, T], \mathbb{R}^n)$, and ω is an arbitrary Riemann–Liouville differentiable vector function, i.e., $\mathbb{D}^{\alpha}_{-\tau^+}\omega$ exists.

The main contributions are stated as follows.

We find a fundamental matrix for homogeneous problem of (3) and then derive its general solution. Next, we derive a special solution for (1) with zero initial condition. Finally, we give a representation of a solution of (1) via the superposition principle.

2. Preliminaries

Let $a, b \in \mathbb{R}, a < b$ and $C((a, b], \mathbb{R}^n)$ be the Banach space of vector-valued continuous function from $(a, b]$ into $\mathbb{R}^n$. Let Θ and I be the zero and identity matrices, respectively.

We recall some definitions and lemmas as follows.

Definition 2.1 (*See [10]*)**.** The Riemann–Liouville derivative of order $0 < \alpha < 1$ for a function $f : [a, \infty) \to \mathbb{R}$ can be written as $(\mathbb{D}^{\alpha}_{a^+}y)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} y(t)dt,\ x > a.$

Definition 2.2 (*See [10]*)**.** The Riemann–Liouville fractional integral of order $0 < \alpha < 1$ for a function $f : [a, \infty) \to \mathbb{R}$ can be written as $(\mathbb{I}^{\alpha}_{a^+}y)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t)dt,\ x > a.$

Definition 2.3 (*See [7, Definition 2.5]*)**.** Set $0 < \alpha < 1$ and $\beta > 0$. Delayed two parameters Mittag-Leffler type matrix $\mathbb{Z}^{B\cdot^{\alpha}}_{\tau,\beta} : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined by

$$\mathbb{Z}^{Bx^{\alpha}}_{\tau,\beta} = \begin{cases} \Theta, & -\infty < x \le -\tau, \\ I\dfrac{(\tau+x)^{\alpha-1}}{\Gamma(\beta)} & -\tau < x \le 0, \\ I\dfrac{(\tau+x)^{\alpha-1}}{\Gamma(\beta)} + B\dfrac{x^{2\alpha-1}}{\Gamma(\alpha+\beta)} + B^2\dfrac{(x-\tau)^{3\alpha-1}}{\Gamma(2\alpha+\beta)} + \cdots + B^k\dfrac{(x-(k-1)\tau)^{(k+1)\alpha-1}}{\Gamma(k\alpha+\beta)}, \\ & (k-1)\tau < x \le k\tau,\ k \in \mathbb{N}^+. \end{cases} \tag{2}$$

Lemma 2.4 (*See [7, Lemma 2.6]*)**.** *For any $(k-1)\tau < x \le k\tau$, $0 \le s \le t$ and $k \in \mathbb{N}^+$ is a fixed number, we have $\int_{(k-1)\tau+s}^x (x-t)^{-\alpha}(t-(k-1)\tau-s)^{k\alpha-1}dt = (x-(k-1)\tau-s)^{(k-1)\alpha}\mathbb{B}[1-\alpha, k\alpha]$, where $\mathbb{B}[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds$ is a Beta function.*

Lemma 2.5 (*See [8, Lemma 2.5]*)**.** *For any $(k-1)\tau < x \le k\tau$ and $k \in \mathbb{N}^+$, we have $\int_{(k-1)\tau}^x (x-t)^{-\alpha}(t-(k-1)\tau)^{(k+1)\alpha-1}dt = (x-(k-1)\tau)^{k\alpha}\mathbb{B}[1-\alpha,(k+1)\alpha]$.*

Lemma 2.6. *Let $(k-1)\tau < x \le k\tau$, $-\tau \le s \le t$ and $k \in \mathbb{N}^+$ is a fixed number, we have $\int_s^x (x-t)^{-\alpha}\mathbb{Z}^{B(t-\tau-s)^{\alpha}}_{\tau,\alpha}dt = \sum_{i=0}^k \int_{i\tau+s}^x (x-t)^{-\alpha}B^i\frac{(t-i\tau-s)^{(i+1)\alpha-1}}{\Gamma(i\alpha+\alpha)}dt.$*

Proof. The proof is similar to [7, Lemma 2.7], so we omit it here. □