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Spectral method for solving the time fractional Boussinesq equation

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ABSTRACT

In this paper, Fourier spectral approximation for the time fractional Boussinesq equation with periodic boundary condition is considered. The space is discretized by the Fourier spectral method and the Crank–Nicolson scheme is used to discretize the Caputo time fractional derivative. Stability and convergence analysis of the numerical method are proven. Some numerical examples are included to testify the effectiveness of our given method. Based on the presented numerical results, the Fourier spectral method is shown to be effective for solving the time fractional Boussinesq equation.

1. Introduction

In recent years, the study of fractional integro differential equations applied to physics and other areas has grown [1–4]. Time fractional Boussinesq equation describing the surface water waves is an important nonlinear equation. Many researchers have paid much attention to the type of equation. Hosseini and Ansari [5] solved the Boussinesq equations with the conformable time-fractional derivative analytically using the well-established modified Kudryashov method. Fractional Lie group method of the time-fractional Boussinesq equation was proposed in [6].

In this paper, the following time fractional Boussinesq equation [5,7] is considered,

$$ {}_0^C D_t^{\alpha} u(x,t) = -u_{xxxx} + u_{xx} + (u^2)_{xx}, \quad 0 < t \le T, \tag{1}$$

$$ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{2}$$

which describes the surface water waves whose horizontal scale is much larger than the depth of the water. $u(x,t)$ is a wave function. We first truncate (1)–(2) into a finite computational domain $[-L, L]$ with periodic

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boundary condition. The operator ${}_0^C D_t^\alpha$ $(1 < \alpha < 2)$ considering the wave diffusion is the time Caputo fractional derivative defined as [8],

$$ {}_0^C D_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{u^{(2)}(s)ds}{(t-s)^{\alpha-1}}. \tag{3} $$

Spectral methods are very powerful tools for treating numerous types of fractional integral and differential equations [9–11] because of their high-order accuracy. So far, most research of the time fractional Boussinesq equation is limited to analytical solutions. In this present paper, we introduce the Fourier spectral method as an efficient alternative approach to solving this equation. Stability and convergence analysis of the numerical method are verified.

2. Numerical method

In order to develop a fully discrete form of time derivative, let τ be the time step size and K be a positive integer with $\tau = T/K$ and $t_k = k\tau$ for $k = 0, 1, \dots, K$. Take $t_{k-1/2} = (\frac{t_{k-1}+t_k}{2})$ and $u^k = u(\cdot, t_k)$. Now we consider (1) on line $(x, t_{k-1/2})$, let $\delta_t u^{k-1/2} = \frac{u^k - u^{k-1}}{\tau}$ and $u^{k-1/2} = \frac{u^{k-1}+u^k}{2}$. The operator ${}_0^C D_t^\alpha$ as in [12] can be discreted by the Crank–Nicolson scheme, that is for any $1 < \alpha < 2$, $k = 1, 2, \dots, K$,

$$ \frac{1}{2}\left({}_0^C D_t^\alpha u^k + {}_0^C D_t^\alpha u^{k-1}\right) = \overline{\nabla}_t^\alpha u^{k-1/2} + R_k^\alpha, \tag{4} $$

and

$$ \overline{\nabla}_t^\alpha u^{k-1/2} = \tau^{1-\alpha}\left[b_0^\alpha \delta_t u^{k-1/2} - \sum_{j=1}^{k-1} (b_{k-1-j}^\alpha - b_{k-j}^\alpha)\delta_t u^{j-1/2} - b_{k-1}^\alpha u_t^0 \right], \tag{5} $$

where $b_j^\alpha = \frac{1}{\Gamma(3-\alpha)}((j+1)^{2-\alpha} - j^{2-\alpha})$, $j = 0, 1, \dots, k-1$ satisfying $b_0^\alpha = \frac{1}{\Gamma(3-\alpha)}$, $\sum_{j=1}^{k} b_{k-j}^\alpha = \frac{k^{2-\alpha}}{\Gamma(3-\alpha)}$, $\sum_{j=1}^{k-1}(b_{k-1-j}^\alpha - b_{k-j}^\alpha) + b_{k-1}^\alpha = \frac{1}{\Gamma(3-\alpha)}$. The truncation error is given by $|R_k^\alpha| \le C \max_{0\le t\le T} \left| \frac{\partial^3 u(x,t)}{\partial t^3} \right| \tau^{3-\alpha}$.

Take $\Omega = (-L, L)$ and $I = (0, T]$. Let $C_{per}^\infty(\Omega)$ be the set of all restrictions onto Ω of all complex-valued, periodic, C^∞-functions on $\mathbb{R}$. For s as a nonnegative real number, let $H_{per}^s(\Omega)$ be the closure of $C_{per}^\infty(\Omega)$ with the norm $\|\cdot\|_s$ and semi-norm $|\cdot|_s$. Note that $H_{per}^0(\Omega) = L_{per}^2(\Omega)$. For a positive integer N, the basis function space $S_N = \text{span}\{e^{i\omega\pi x/L} : -\frac{N}{2} \leqslant \omega \leqslant \frac{N}{2} - 1\}$ can be set. For any function $u(x,t)$, we have

$$ u_N(t) = \sum_{\omega=-N/2}^{N/2-1} \hat{u}_\omega(t) e^{i\omega\pi x/L}, \tag{6} $$

where the Fourier coefficients are arranged as $\hat{u}_\omega = (u, e^{i\omega\pi x/L}) = \frac{1}{2L}\int_\Omega u e^{-i\omega\pi x/L} dx$. Based on the above results, the fully discrete Fourier spectral approximation for (1)–(2) has a modified scheme as follows: find $u_N^k \in S_N$,

$$ (\overline{\nabla}_t^\alpha u_N^{k-1/2}, v) = -(\partial_{xxxx} u_N^{k-1/2}, v) + (\partial_{xx} u_N^{k-1/2}, v) + \frac{1}{2}(\partial_{xx}(F(u_N^k) + F(u_N^{k-1})), v), \quad \forall v \in S_N, \tag{7} $$

and

$$ u_N^0 = \Pi_N u_0(x), \quad (u_N^0)_t = \Pi_N u_1(x), \tag{8} $$

where $F(u) = u^2$, Π_N: $L_{per}^2(\Omega) \to S_N$ is the L^2-orthogonal projection [11].