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# On the optimal convergence rates of Chebyshev interpolations for functions of limited regularity

Shuhuang Xiang[1]

**Abstract**. In this paper, some new asymptotic formulas on the decay of the coefficients of the Chebyshev expansions are presented and refined estimates on the asymptotics for functions of limited regularities are derived. Together with these, the optimal convergence rates of the polynomial interpolations in Chebyshev points are deduced.

**Keywords.** asymptotic, Chebyshev expansion, convergence rate, interpolation.

## 1 Introduction

Polynomial approximation is used as the basic means of approximation of functions that are difficult to be computed, the solutions of differential and integral equations, and application in numerical integration. Weierstrass [13] in 1885 proved the well known result that every continuous function $f(x)$ in $[-1,1]$ can be uniformly approximated as closely as desired by a polynomial function.

Among the set of polynomials $\mathcal{P}_n$ of degree less than or equal to $n$, the best polynomial approximation in $\mathcal{P}_n$ to $f(x)$, denoted by $p_n^*$, satisfies

$$E_n(f;[-1,1]) = \|f - p_n^*\|_\infty = \min_{p\in\mathcal{P}_n} \|f-p\|_\infty.$$

Let $\omega(\delta)$ denote the modulus of continuity of $f(x)$ on $[-1,1]$ defined for $\delta>0$ by

$$\omega(\delta) = \sup_{|x_1-x_2|\le\delta,\, x_1,x_2\in[-1,1]} |f(x_1)-f(x_2)|.$$

It is well known that

(i) ([7, Jackson's theorem]) if $f\in C[-1,1]$, then $E_n(f;[-1,1]) = \max_{-1\le x\le 1}|f(x)-p_n^*| \le 6\omega\left(\frac{1}{n}\right)$;

(ii) ([7, p. 22]) if $f\in \mathrm{lip}_{\mathrm{K}}\gamma$ on $[-1,1]$, then $E_n(f;[-1,1]) = \max_{-1\le x\le 1}|f(x)-p_n^*| \le 6Kn^{-\gamma}$; and if $f'\in C[-1,1]$, then $E_n(f;[-1,1]) \le 6E_{n-1}(f';[-1,1])n^{-1}$.

From the above results, we see that if $f(x)$ has a $k$th derivative on $[-1,1]$, then, $E_n(f;[-1,1]) \le \frac{6^k}{n(n-1)\cdots(n-k+1)}\omega_k\left(\frac{1}{n-k}\right) \le \frac{c}{n^k}\omega_k\left(\frac{1}{n-k}\right)$ holds for $n>k$, where $\omega_k$ is the modulus of continuity of $f^{(k)}$ and $c = 6^{k+1}e^k(1+k)^{-1}$ [7, p. 23]. In particular, if $f^{(k)}\in \mathrm{lip}_{\mathrm{K}}\gamma$, then $E_n(f;[-1,1]) \le \frac{cK}{n^k(n-k)^\gamma}$.

However, the computation of the best approximation polynomial $p_n^*$ is a NP-hard problem [9]. Thus, polynomial interpolation is a powerful tool to approximate functions that are difficult to be computed.

There is a well developed theory that quantifies the convergence of the interpolation polynomial $L_n[f]$ by the Lebesgue constant $\Lambda_n$ in a given set of points (Trefethen [9])

$$\big\|L_n[f]-f\big\|_\infty \le (1+\Lambda_n)\big\|p_n^*-f\big\|_\infty. \tag{1.1}$$

Erdös [4] proved that $\Lambda_n \ge \frac{2}{\pi}\log n + C$ for some constant $C$ for arbitrary given set of points. Bernstein [1] and Ehlich and Zeller [3] showed that for the Chebyshev points of first or second kind $\Lambda_n \sim \frac{2}{\pi}\log n$.

It has been known that the interpolation polynomial in the Chebyshev points of the first or second kind does not suffer from the Runge phenomenon [9]. In particular, these polynomials can be efficiently

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