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The properties of the solutions of the incompressible flows on an exterior domain

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ABSTRACT

In this paper, we want to see the properties of the smooth solutions $\mathbf{u}$ of the incompressible flows on an exterior domain Ω of R^2. Specially, when the vorticity $\omega = \nabla \times \mathbf{u}$ has a bounded support, with suitable conditions we will show that there exists a constant $C(p, q)$ such that $\| \mathbf{u} \|_{L^p(\Omega)} \leq C \| \mathbf{u} \|_{L^q(\Omega)}$ for $1 < p \leq q \leq \infty$.

1. Introduction

It is also well known that if a domain $\Omega \subset R^n$ is a bounded set then one has

$$|\Omega|^{-\frac{1}{p}} \|u\|_{L^p(\Omega)} \leq |\Omega|^{-\frac{1}{q}} \|u\|_{L^q(\Omega)}, \quad \text{for} \quad 1 \leq p < q \leq \infty, \tag{1.1}$$

where $u \in L^q(\Omega)$. But, generally above inequality is no longer true on an unbounded domain unless u has a bounded support. In this paper, we want to see if we can obtain similar inequality with (1.1) for the solutions of the incompressible flows on unbounded domains when the support of the vorticity is a bounded set. This paper was motivated to study the long time behavior of the solutions for the two dimensional Euler equations. Therefore, for the future work, with the results of this paper we will study the asymptotic behavior of the solutions for the two dimensional Euler equations on an exterior domain. This idea was first initiated by the paper Marchioro [1] that the vorticity $\omega(t)$ of the solutions $\mathbf{u}$ of the two dimensional Euler equations has a bounded support for all finite time when an initial vorticity ω_0 has a bounded support.

2. Main results

Theorem 2.1. *Let $\mathbf{u}(x)$ be the globally smooth enough vector function on an exterior domain Ω of R^2 where the curl $\nabla \times \mathbf{u}$ has a bounded support contained in a ball $B(0, K)$. We also assume that $\nabla \cdot \mathbf{u}(x) = 0$*

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for all $x \in \Omega$ and $\mathbf{u}$ *vanishes sufficiently rapidly as $|x| \nearrow \infty$. Then, there exists a constant $C(p,q)$ such that*

$$\|\mathbf{u}\|_{L^p(\Omega)} \leq \ CK^{\frac{2}{p}-\frac{2}{q}}\ \|\mathbf{u}\|_{L^q(\Omega)}, \quad \textit{for} \ \ 1 < p < q \leq \infty. \tag{2.1}$$

Proof. Let us consider a radial cut-off function $\sigma \in C^\infty(R^2)$ such that

$$\sigma(x) = \sigma(|x|) = 0 \ \text{ if } |x| < 1, \quad \sigma(x) = \sigma(|x|) = 1 \ \text{ if } |x| > 2,$$

and $0 \leq \sigma(x) = \sigma(|x|) \leq 1$ for $1 < |x| < 2$. And, for each $R > 0$, we define

$$\sigma\left(\frac{|x|}{R}\right) = \sigma_R(|x|) \in C^\infty(R^2).$$

One note that $\|\nabla \sigma_R\|_{L^2(R^2)}$ is bounded by some constant η which is independent to R. Now, we define new radial cut-off function $\phi_r(x) \in C_0^\infty(R^2)$,

$$\phi_r(x) = \phi_r(|x|) = \sigma\left(\frac{|x|}{K}\right)\left[1 - \sigma\left(\frac{|x|}{r}\right)\right], \quad \text{for large } r > 3K.$$

And, by the definition of ϕ_r, we can set as

$$\mathbf{v}(x) = \int_{R^2} N(x-y)[\phi_r(y)(\nabla \times \mathbf{u})(y)]dy = \int_{\Omega} N(x-y)[\phi_r(y)(\nabla \times \mathbf{u})(y)]dy, \tag{2.2}$$

where N be the fundamental function of $-\Delta$ on R^2. We consider $\mathbf{z}$ as $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, 0)$ to calculate $\nabla \times \mathbf{z}$ and denote $\nabla \times \mathbf{z}$ as $(\nabla \times \mathbf{z}) \cdot e_3 = \frac{\partial \mathbf{z}_2}{\partial x} - \frac{\partial \mathbf{z}_1}{\partial y}$ without any more comments. Then, since $\nabla \cdot \mathbf{u} = 0$, due to

$$\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - \nabla^2 F \quad \text{and} \quad \phi_r(\nabla \times \mathbf{u}(y)) = \nabla \times (\phi_r \mathbf{u}) - (\nabla \phi_r) \times \mathbf{u},$$

we have from (2.2) that

$$\begin{aligned} \nabla \times \mathbf{v} &= \nabla \times \int_{R^2} N(x-y)(\nabla \times \phi_r \mathbf{u}(y))dy - \nabla \times \int_{R^2} N(x-y)(\nabla \phi_r) \times \mathbf{u}(y)dy \\ &= \phi_r \mathbf{u} - \nabla \times [N * ((\nabla \phi_r) \times \mathbf{u})] + \nabla[N * (\mathbf{u} \cdot \nabla \phi_r)]. \end{aligned}$$

Since $\nabla \times \mathbf{v} = 0$ from the definition of ϕ_r, we obtain

$$\phi_r \mathbf{u} = \nabla \times [N * ((\nabla \phi_r) \times \mathbf{u})] - \nabla[N * (\mathbf{u} \cdot \nabla \phi_r)]. \tag{2.3}$$

Now, we remind the Sobolev's Theorem (Theorem 9.3 in page 91 of [2]): *For $\Psi(x) = \int_{R^n} K(x,y) f(y)dy$ where $K(x,y) = \frac{1}{|x-y|^\lambda}$, $\lambda > 0$, if $f \in L^q(R^n)$, $1 < q < \infty$ and $\lambda > n(1 - \frac{1}{q})$ then we have $c(q,n,\lambda)$ such that*

$$\|\Psi\|_{L^s} \leq c\|f\|_{L^q}, \qquad \text{for } \frac{1}{s} = \frac{\lambda}{n} + \frac{1}{q} - 1. \tag{2.4}$$

In this proof we denote

$$\begin{aligned} &\Omega_0 = \{x \in R^2 \ : \ |x| \leq K\}, \quad \Omega_1 = \{x \in R^2 \ : \ K \leq |x| \leq 2K\}, \\ &\Omega_2 = \{x \in R^2 \ : \ |x| > 2K\}, \quad \Omega_3 = \Omega_0 \cup \Omega_1. \end{aligned}$$

Since $\|\nabla \phi_r\|_{L^2(R^2)} \leq 2\eta$ for some $\eta > 0$, due to (2.3) and (2.4), we obtain

$$\|\phi_r \mathbf{u}\|_{L^p(R^2)} \leq C_p \left(\|\mathbf{u}\|_{L^p(\Omega_1)} + \|\mathbf{u}\|_{L^p(\Omega_r)}\right), \quad \text{for } \ 2 < p < \infty, \tag{2.5}$$