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A new analytical formula for the wave equations with variable coefficients☆

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ABSTRACT

This article presents a new analytical formula for the Cauchy problem of the wave equation with variable coefficients, which is a much simpler solution than that given by the Poisson formula. The derivation is based on the variation-of-constants formula and the theory of pseudodifferential operator. The formula is applied to an example to illustrate the feasibility.

1. Introduction

The study of partial differential equations can date back to the 18th century in the context of the movements of models in the physics of continuous media [1]. The history of partial differential equations began with the one-dimensional wave equation $u_{tt} = u_{xx}$ introduced by d'Alembert as a model of a vibrating string [2]. Giving its initial displacement $u(x, 0) = \varphi(x)$ and velocity $u_t(x, 0) = \psi(x)$, it is known to all that an explicit solution to the Cauchy, or initial value, problem of the wave equation is

$$u(x,t) = \frac{1}{2}(\varphi(x-t) + \varphi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)d\xi.$$

For the Cauchy problem of the wave equation in higher dimensions, the derivation of the solution is very technical. Various methods have been utilized to solve the Cauchy problem for higher dimensional wave

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equation. For example, the Poisson formula for $n = 3$ is proved by the method of spherical means and the solution for $n = 2$ is derived by the Hadamard method of descent [3]. A formal series solution has been obtained separately using a classical power series in [4] and the Adomian decomposition method in [5]. A direct derivation of the spherical means solution for arbitrary dimension n is presented using Fourier transform in [6]. In [7–9], formulating the wave equation as an abstract ordinary differential equations and applying Duhamel principle, the authors gave the so-called operator–variation-of-constants formula for semilinear wave equations with constant coefficients under different types of boundary conditions. Motivated by [7,8], the use of variation-of-constants formula and the interpretation of differential operator with variable coefficients as pseudodifferential operator are the essential ingredients in this paper.

Consider the Cauchy problem for the following linear homogeneous wave equation with variable coefficients in $\mathbb{R}^n$

$$\begin{cases} u_{tt} - Lu = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $L = \sum_{i=1}^{n}\sum_{1=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$ is a second order linear differential operator with variable coefficients satisfying that the matrix of coefficients $A = (a_{ij}(x))_{n\times n}$ is symmetric and positive semi-definite for all $x \in \mathbb{R}^n$. It is noted that the form of wave equation (1) is nearly the same as the following oscillatory system of ordinary differential equations with the right-hand side $f \equiv 0$.

$$\begin{cases} q_{tt}(t) + Mq(t) = f(q(t)), \\ q(0) = q_0, q_t(0) = p_0, \end{cases} \tag{2}$$

where $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$ and M is a symmetric and positive semi-definite matrix.

In recent years, there has been an enormous advance in numerically solving the oscillatory system (2). Many useful approaches to constructing Runge–Kutta–Nyström (RKN)-type integrators have been proposed (see, e.g. [10–14]). Taking account of the special structure introduced by the linear term Mq, Wu et al. [14] formulated a standard form of the multidimensional extended RKN (ERKN) integrators. The ERKN integrators naturally integrate exactly the unperturbed linear equation $q_{tt} + Mq = 0$, which has the same form as that of the equation concerned in the paper. The following variation-of-constants formula is established in [13] which is essential for ERKN integrators. It gives the exact solution and its derivative for the oscillatory system (2):

$$\begin{cases} q(t) = \phi_0(t^2M)q_0 + t\phi_1(t^2M)p_0 + \displaystyle\int_0^t (t-s)\phi_1((t-s)^2M)f\big(q(s)\big)ds, \\ q_t(t) = -tM\phi_1(t^2M)q_0 + \phi_0(t^2M)p_0 + \displaystyle\int_0^t \phi_0((t-s)^2M)f\big(q(s)\big)ds, \end{cases} \tag{3}$$

where the analytical functions $\phi_j(\cdot), j = 0, 1, \ldots$ are defined as

$$\phi_j(x) := \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{(2i+j)!}, \quad j = 0, 1, \ldots. \tag{4}$$

For more details on the ERKN integrator, we refer the reader to [15–19].

Letting the right-hand-side function $f \equiv 0$, we have that the exact solution of the linear homogeneous ordinary differential equations

$$\begin{cases} q_{tt}(t) + Mq(t) = 0, \\ q(0) = q_0, q_t(0) = p_0 \end{cases} \tag{5}$$