Applied Mathematics Letters 84 (2018) 160–167

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Linearized compact ADI schemes for nonlinear time-fractional Schrödinger equations[☆]

Xiaoli Chen[a,b], Yana Di[c,d], Jinqiao Duan[b,e], Dongfang Li[a,f,*]

[a] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China*
[b] *Center for Mathematical Sciences, Huazhong University of Science and Technology, Wuhan 430074, China*
[c] *LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China*
[d] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, China*
[e] *Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA*
[f] *Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, China*

ARTICLE INFO

Article history:
Received 24 February 2018
Received in revised form 5 May 2018
Accepted 5 May 2018
Available online 29 May 2018

Keywords:
Nonlinear time fractional Schrödinger equation
Compact ADI scheme
Convergence

ABSTRACT

This paper is concerned with the construction and analysis of linearized numerical methods for solving the two-dimensional nonlinear time fractional Schrödinger equations. By adding different correction terms, two linearized compact alternating direction implicit (ADI) methods are proposed. Convergence of the proposed methods is obtained. Numerical results are presented to verify the accuracy and efficiency of the proposed schemes.

1. Introduction

In this paper, we present two linearized compact ADI schemes for solving the two-dimensional nonlinear time fractional Schrödinger equations (TFSEs):

$$\begin{cases} \mathrm{i}\,{}_0^C D_t^\alpha u + \triangle u = f(|u|^2)u, & \text{in } \Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega \times \{0\}, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \end{cases} \tag{1.1}$$

☆ This work is supported by NSFC (Grant Nos. 11771162, 11771437, 91630208, 91641107, and 11571128.)

* Corresponding author at: School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China.

E-mail address: dfli@mail.hust.edu.cn (D. Li).

https://doi.org/10.1016/j.aml.2018.05.007
0893-9659/

where $\Omega \subset \mathbb{R}^2$ is a bounded and smooth domain, $u_0(x)$ is a known initial condition, $f \in C^2(\mathbb{R})$, and ${}_0^C D_t^\alpha (0 < \alpha < 1)$ represents the Caputo fractional derivative, defined by

$$ {}_0^C D_t^\alpha u(\cdot, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\cdot, s)}{\partial s} \frac{1}{(t-s)^\alpha} ds, \tag{1.2} $$

where $\Gamma(\cdot)$ denotes the usual gamma function.

TFSEs are widely used to describe many phenomena, such as the non-Markovian evolution of a free particle in quantum physics [1], the fractional dynamics in quantum mechanics [2], the fractional Planck quantum energy relation [3] and so on. In recent years, it has become an indispensable part to learn the evolutionary of the models by the numerical simulation. For example, Mohebbi et al. [4] applied the meshless method to solve the TFSEs. Bhrawy and Abdelkawy [5] investigated the spectral collocation approximations. Li et al. studied the finite element approximations [6]. For more numerical investigations of TFSEs and the related models, we refer the readers to [7–10] for an incomplete list of references. However, the computational cost may be very high by directly solving high-dimensional nonlinear TFSEs.

In this study, two linearized numerical methods are developed to investigate the two dimensional nonlinear TFSEs. The schemes are constructed as follows. The time fractional derivative is approximated by using the L1-method, and the second order spatial derivatives are approximated by using the compact finite difference scheme. Together with the different correction terms and ADI method, the original equations are split into two separate one-dimensional problems. The computational cost can be reduced. The error estimates of the fully discrete numerical schemes are obtained. Numerical examples are presented to confirm the theoretical results. The remainder of the paper is organized as follows. In Section 2, the fully discrete numerical schemes are proposed. In Section 3, convergence of the proposed schemes is obtained. In Section 4, numerical results are given to verify the theoretical analysis.

2. The derivation of the linearized compact difference schemes

This section is concerned with the construction of the linearized numerical methods for two dimensional nonlinear TFSEs, whose solutions are assumed to be sufficiently regular. Without loss of generality, we set $\Omega = (0,1) \times (0,1)$ here and below.

Let $\tau = \frac{T}{N}$ be the temporal step size and $h_x = \frac{1}{M_x}$, $h_y = \frac{1}{M_y}$ be spatial step sizes, where N, M_x and M_y are given positive integers. Denote $\Omega_\tau = \{t_n | t_n = n\tau,\ 0 \le n \le N\}$, $\Omega_h = \bar{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$, where $\bar{\Omega}_h = \{(x_j, y_k) | x_j = jh_x,\ 0 \le j \le M_x;\ y_k = kh_y,\ 0 \le k \le M_y\}$. Let $\mathcal{V} = \{v_{jk}^n |\ v_{0k}^n = v_{j0}^n = v_{M_x k}^n = v_{jM_y}^n = 0,\ j = 0, 1, \ldots, M_x,\ k = 0, 1, \ldots, M_y,\ n = 0, 1, \ldots, N\}$ be grid function space defined on $\Omega_h \times \Omega_\tau$. Define

$$ D_\tau^L v_{jk}^n = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[v_{jk}^n + \sum_{m=1}^{n-1} (a_{n-m} - a_{n-m-1}) v_{jk}^m - a_{n-1} v_{jk}^0 \right], \quad \text{with } a_l = (l+1)^{1-\alpha} - l^{1-\alpha},\ \ l \ge 0, $$

$$ \delta_t v_{jk}^{n-\frac{1}{2}} = \frac{1}{\tau}(v_{jk}^n - v_{jk}^{n-1}), \delta_x^2 v_{jk}^n = \frac{1}{h_x^2}(v_{j-1,k}^n - 2v_{jk}^n + v_{j+1,k}^n),\ \delta_y^2 v_{jk}^n = \frac{1}{h_y^2}(v_{j,k-1}^n - 2v_{jk}^n + v_{j,k+1}^n), $$

$$ \delta_x v_{j-\frac{1}{2},k}^n = \frac{1}{h_x}(v_{jk}^n - v_{j-1,k}^n),\ \delta_y v_{j,k-\frac{1}{2}}^n = \frac{1}{h_y}(v_{jk}^n - v_{j,k-1}^n), $$

$$ (u^n, v^n) = h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} u_{jk}^n \overline{v}_{jk}^n,\ \|v^n\| = \sqrt{(v^n, v^n)}, $$

$$ \mathcal{H}_x v_{jk}^n = \begin{cases} (1 + \frac{h_x^2}{12}\delta_x^2) v_{jk}^n, & 1 \le j \le M_x - 1, \quad 0 \le k \le M_y, \\ v_{jk}^n, & j = 0 \ \text{ or } \ M_x, \quad 0 \le k \le M_y, \end{cases} $$