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Global solutions to isothermal system in a divergent nozzle with friction

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ABSTRACT

In this paper, we remove the restriction $A'(x) \geq 0$ in the paper 'Lu (2011)', the restriction $z_0(x) \leq 0$ or $w_0(x) \leq 0$ in the paper 'Klingenberg and Lu (1997)', and obtain the global existence of entropy solutions to the isothermal gas dynamics system in a divergent nozzle with friction.

1. Introduction

In this paper we studied the global entropy solutions for the Cauchy problem of isentropic gas dynamics system in a divergent nozzle with a friction, whose physical phenomena called "choking or choked flow", occurs in the nozzle

$$\begin{cases} \rho_t + (\rho u)_x = -\dfrac{a'(x)}{a(x)}\rho u \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\dfrac{a'(x)}{a(x)}\rho u^2 - \alpha \rho u|u|, \end{cases} \tag{1.1}$$

with bounded initial date

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0, \tag{1.2}$$

where ρ is the density of gas, u the velocity, $P = P(\rho)$ the pressure, $a(x)$ is a slowly variable cross section area at x in the nozzle and α denotes a friction constant. For the polytropic gas, P takes the special form $P(\rho) = \frac{1}{\gamma}\rho^{\gamma}$, where $\gamma > 1$ is the adiabatic exponent and for the isothermal gas, $\gamma = 1$. System (1.1) is of

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interest because resonance occurs. This means there is a coincidence of wave speeds from different families of waves (See [1–4] and the references cited therein for the details).

By simple calculations, two eigenvalues of system (1.1) are

$$\lambda_1 = \frac{m}{\rho} - \sqrt{P'(\rho)}, \quad \lambda_2 = \frac{m}{\rho} + \sqrt{P'(\rho)} \tag{1.3}$$

with corresponding Riemann invariants

$$z(u,v) = \int_0^\rho \frac{\sqrt{P'(s)}}{s} ds - \frac{m}{\rho}, \quad w(u,v) = \int_0^\rho \frac{\sqrt{P'(s)}}{s} ds + \frac{m}{\rho}, \tag{1.4}$$

where $m = \rho u$.

When $a'(x) = 0$, (1.1) is the river flow equations, a shallow-water model describing the vertical depth ρ and mean velocity u, where $\alpha\rho u|u|$ corresponds physically to a friction term, and its global, bounded solutions were obtained in [5].

When $\alpha = 0$, i.e., the nozzle flow without friction, system (1.1) was well studied in [6–8] for the polytropic gas ($\gamma > 1$), and in [9] for the isothermal gas ($\gamma = 1$). In [9], a strong technique condition $A'(x) \geq 0$ was imposed when the author used the maximum principle to study the positive lower bound of the density ρ (the bound depends on the viscosity coefficient ε).

Since the super-linear source terms in (1.1), when we prove the global existence, the main difficulty is to obtain L^∞ estimates, of viscosity solutions, independent of the viscosity perturbation constant ε. With the help of the condition $z_0(x) \leq 0$ or $w_0(x) \leq 0$, the L^∞ bound of $(\rho^\varepsilon, m^\varepsilon)$ was obtained in [10,11] for the polytropic gas $\gamma > 1$.

Since the case of $\gamma = 1$ is different from that of $\gamma > 1$, in this paper, we remove the conditions $z_0(x) \leq 0$ or $w_0(x) \leq 0$ in [10,11], and $A'(x) \geq 0$ for the isothermal case $P(\rho) = \rho$ in [9], and prove the global existence of weak solutions for the Cauchy problem (1.1)–(1.2) for general bounded initial date. The main result is given in the following

Theorem 1.1. *Let $P(\rho) = \rho, 0 < a_L \leq a(x) \leq M$ for x in any compact set $x \in (-L, L), A(x) = -\frac{a'(x)}{a(x)} \in C^1(\mathbb{R})$ and $|A(x)| \leq M$, where M, a_L are positive constants, but a_L could depend on L. Then the Cauchy problem (1.1)–(1.2) has a bounded weak solution (ρ, u) which satisfies system (1.1) in the sense of distributions and*

$$\int_0^\infty \int_{-\infty}^\infty \eta(\rho,m)\phi_t + q(\rho,m)\phi_x + A(x)(\eta(\rho,m)_\rho \rho u + \eta(\rho,m)_m \rho u^2)\phi dxdt \geq 0, \tag{1.5}$$

where (η, q) is a pair of entropy–entropy flux of system (1.1), η is convex, and $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+ - \{t = 0\})$ is a positive function.

Remark 1.2. The global existence of symmetrical weak solutions of the isothermal gas dynamics system (1.1) without a friction in the Lagrangian coordinates was well studied in [12–14] by using the Glimm scheme method [15,16].

Remark 1.3. The homogeneous case of isothermal system (1.1) ($a'(x) = 0, \alpha = 0$) in the Euler coordinates was studied in [17] by using the compensated compactness theory [18].

2. Proof of Theorem 1.1

Let $v = \rho a(x)$, then we may rewrite (1.1) as

$$\begin{cases} v_t + (vu)_x = 0 \\ (vu)_t + (vu^2 + v)_x + A(x)v + \alpha vu|u| = 0. \end{cases} \tag{2.1}$$