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Concentration for blow-up solutions of semi-relativistic Hartree equations of critical type

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This paper concerns the semi-relativistic Hartree equation

$$i\partial_t u = \sqrt{-\Delta + m^2}u - (|\cdot|^{-1} * |u|^2)u$$

in $\mathbb{R}^3$. We prove the concentration results for finite time blow-up solutions with general $H_x^{1/2}(\mathbb{R}^3)$ data, and show the relation between the concentration rate and the blow-up order.

1. Introduction

Consider the Cauchy problem of the semi-relativistic Hartree equation with L^2-critical nonlinearity

$$\begin{cases} i\partial_t u = \sqrt{-\Delta + m^2}u - (|\cdot|^{-1} * |u|^2)u, \quad x \in \mathbb{R}^3, \\ u(0,x) = u_0(x) \in H_x^{1/2}(\mathbb{R}^3). \end{cases} \tag{1}$$

Here u is a complex valued function defined on some spatial-time slab $[0,T) \times \mathbb{R}^3$, $m \geq 0$ is the mass of a particle, $\sqrt{-\Delta + m^2}$ is defined by the symbol $\sqrt{|\xi|^2 + m^2}$ in Fourier space, and $*$ stands for convolution in $\mathbb{R}^3$.

The equation in (1) has been derived as a model describing the mean field dynamics of boson stars [1]. The well-posedness problems in the energy space $H^{\frac{1}{2}}$ and lower regularity Sobolev spaces $H^s (s < \frac{1}{2})$ were considered in [2–4]. Assume momentarily $m = 0$, then the equation and the L^2-norm of the initial datum are invariant under the scaling $u_\rho(t,x) = \rho^{3/2}u(\rho t, \rho x)$. For an $H_x^{1/2}(\mathbb{R}^3)$ solution u, by a regularization method,

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it satisfies two conservation laws [4]:

$$M(u(t)) := \int_{\mathbb{R}^3} |u(t,x)|^2 \, \mathrm{d}x \equiv M(u(0)), \tag{2}$$

$$E_m(u(t)) := \frac{1}{2}\int_{\mathbb{R}^3} \bar{u}(t,x)\sqrt{-\Delta+m^2}u(t,x)\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} (|\cdot|^{-1} * |u|^2)|u(t,x)|^2 \, \mathrm{d}x \equiv E_m(u(0)). \tag{3}$$

Here $\bar{u}$ denotes the conjugate of u.

From [4], (1) is locally well-posed in $H_x^{1/2}(\mathbb{R}^3)$ and the blow-up alternative is either the maximal existence time $T = \infty$ or $T < \infty$, and $\lim_{t\nearrow T}\|u(t)\|_{\dot{H}_x^{1/2}(\mathbb{R}^3)} = \infty$. Moreover, by considering the minimizer for the functional

$$G[u] = \frac{\|(-\Delta)^{1/4}u\|^2_{L^2_x}\|u\|^2_{L^2_x}}{\int_{\mathbb{R}^3}(|\cdot|^{-1} * |u|^2)|u(x)|^2 \, \mathrm{d}x},$$

one infers that if the initial datum u_0 satisfies $\|u_0\|_{L^2_x} < \|Q\|_{L^2_x}$, then the corresponding solution u is global in $H_x^{1/2}(\mathbb{R}^3)$. Here Q is the minimizer for $G[u]$, and is the unique positive, radial, and non-increasing smooth solution (ground state) to the elliptic equation

$$(-\Delta)^{1/2}Q + Q = (|\cdot|^{-1} * |Q|^2)Q. \tag{4}$$

The author proved in [4] that the condition $\|u_0\|_{L^2_x} < \|Q\|_{L^2_x}$ is irrelevant with parameter m. On the other hand, $\|u_0\|_{L^2_x} < \|Q\|_{L^2_x}$ is optimal in the sense that there exist finite time blow-up solutions with $\|u_0\|_{L^2_x} > \|Q\|_{L^2_x}$. Indeed, from [5], for all $N > \|Q\|_{L^2_x}$ there exists u_0 with $\|u_0\|_{L^2_x} = N$ and such that $E_m(u_0) < 0$. By [6], Fröhlich and Lenzmann showed by a variance estimate that finite time blow-up solution exists provided that the initial datum satisfies $E_m(u_0) < 0$. These lead to a problem whether concentration occurs at blow-up time if a solution blows up. The concentration phenomena for nonlinear Schrödinger equations have been well studied, see e.g., [7–9].

The purpose of this work is to investigate concentration phenomena for finite time blow-up solutions of (1). The first main result is the following.

Theorem 1. *Let u be the solution of (1) and u blows up at finite time $T > 0$. Assume $\lambda(t) > 0$ obeying $\sqrt{\lambda(t)}\|u(t)\|_{\dot{H}_x^{1/2}(\mathbb{R}^3)} \to \infty$ as $t \nearrow T$. Then, there exists $x(t) \in \mathbb{R}^3$ such that*

$$\liminf_{t\nearrow T} \int_{|x-x(t)|\le\lambda(t)} |u(t,x)|^2 \, \mathrm{d}x \ge \|Q\|^2_{L^2_x(\mathbb{R}^3)}. \tag{5}$$

We refer to $\lambda(t)$ as blow-up order. We remark that if $m = 0$, then by setting $u_\rho(\tau,x) = \rho(t)^{3/2}u(t+\rho(t)\tau, \rho x)$ with $\rho(t)\|u(t)\|^2_{\dot{H}_x^{1/2}(\mathbb{R}^3)} = 1$, we see $\|u_\rho(0)\|_{\dot{H}_x^{1/2}(\mathbb{R}^3)} = 1$. From local theory in $H_x^{1/2}(\mathbb{R}^3)$, there exists $\tau_0 > 0$ such that u_ρ is defined on $[0,\tau_0]$. Since u is defined in $[0,T)$, we have $t + \rho(t)\tau_0 \le T$, which implies that

$$\|u(t)\|_{\dot{H}_x^{1/2}(\mathbb{R}^3)} \ge C(T-t)^{-1/2}.$$

A concentration result which does not concern the concentration rate has been proven in [10]. It reads: There exists $y(t) \in \mathbb{R}^3$ such that

$$\liminf_{t\nearrow T} \int_{|x-y(t)|\le R} |u(t,x)|^2 \, \mathrm{d}x \ge \|Q\|^2_{L^2_x(\mathbb{R}^3)}, \quad \forall R > 0.$$