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A note on one-dimensional time fractional ODEs

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Abstract

In this note, we prove or re-prove several important results regarding one dimensional time fractional ODEs following our previous work [4]. Here we use the definition of Caputo derivative proposed in [8, 10] based on a convolution group. In particular, we establish generalized comparison principles consistent with the new definition of Caputo derivatives. In addition, we establish the full asymptotic behaviors of the solutions for $D_c^\gamma u = Au^p$. Lastly, we provide a simplified proof for the strict monotonicity and stability in initial values for the time fractional differential equations with weak assumptions.

1 Introduction

The fractional calculus in time has been used widely in physics and engineering for memory effect, viscoelasticity, porous media etc [5, 7, 2, 1, 9]. There is a huge amount of literature discussing time fractional differential equations. For instance, one can find some results in [3, 2] using the classic Caputo derivatives. In this paper, we study the following time fractional ODE:

$$D_c^\gamma u = f(t, u),\ u(0) = u_0, \tag{1.1}$$

for $\gamma \in (0, 1)$ and f measurable. Here $D_c^\gamma u$ is the generalized Caputo derivative introduced in [8, 10]. As we will see later, this generalized definition is theoretically more convenient, since it allows us to take advantage of the underlying group structure.

As in [8], we use the following distributions $\{g_\beta\}$ as convolution kernels for $\beta \in (-1, 0)$:

$$g_\beta(t) = \frac{1}{\Gamma(1+\beta)} D\left(\theta(t)t^\beta\right).$$

Here $\theta(t)$ is the standard Heaviside step function, $\Gamma(\cdot)$ is the gamma function, and D means the distributional derivative on $\mathbb{R}$. Indeed, g_β can be defined for $\beta \in \mathbb{R}$ (see [8]) so that $\{g_\beta : \beta \in \mathbb{R}\}$ forms a convolution group. In particular, we have

$$g_{\beta_1} * g_{\beta_2} = g_{\beta_1+\beta_2}. \tag{1.2}$$

Here since the support of g_{β_i} $(i = 1, 2)$ is bounded from left, the convolution is well-defined. Now we are able to give the generalized definition of fractional derivatives:

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