Accepted Manuscript

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PII: S0893-9659(18)30087-9
DOI: https://doi.org/10.1016/j.aml.2018.03.022
Reference: AML 5470

To appear in: *Applied Mathematics Letters*

Received date : 22 February 2018
Accepted date : 19 March 2018

ON STATE DEPENDENT NON-LOCAL CONDITIONS

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ABSTRACT. We introduce a new type of non-local conditions, which we call state dependent non-local conditions, and we study existence and uniqueness of solutions for abstract differential equation subjected to this class of conditions. The non-local condition proposed generalizes several types of non-local conditions studied in the literature. Some examples are given to illustrate our theory.

1. Introduction

In this paper we introduce and study existence and uniqueness of solutions for a new class of abstract differential equations with non-local conditions of the form

$$u'(t) = Au(t) + F(t, u(\gamma(t))), \quad t \in [0, a], \tag{1.1}$$

$$u(0) = H(\sigma(u), u) \in X, \tag{1.2}$$

where $A : D(A) \subset X \mapsto X$ is the generator of an analytic semigroup of linear operators $(T(t))_{t\geq 0}$ on a Banach space $(X, \| \cdot \|)$ and $F(\cdot), \sigma(\cdot), \gamma(\cdot)$ and $H(\cdot)$ are suitable continuous functions.

Differential equations with non-local conditions arise naturally in applications. In mathematical modeling of real processes, non-local conditions can be seen as feedback controls by which a particular qualitative property or magnitude of the solution along its evolution equals its initial state. The main novelty of this work, is the non-local condition (1.2) which is motivated by theory and applications and it permits us to generalize several types of non-local conditions considered in the literature. To illustrate our general idea, we note that the condition $u_0 = \alpha u(\sigma(u))$, where $\sigma \in C(C([0, a]; X); [0, a])$ is a function defined by an equation of the form $\int_0^{\sigma(u)} (\frac{C_1}{C_2 + \|u(s)\|^2_{L^2(\Omega)}} + C_3) ds = C_4$ (a threshold condition, see [11, 16]), is a natural prototype of the type of non-local condition considered in our paper.

Concerning the literature on differential equations with non-local conditions, we cite the pioneer works [6, 7] for ODEs, and the book [4] and the papers [1, 2, 3, 12, 8, 9, 10, 5, 13, 15] for abstract differential equations and partial differential equations.

Existence and uniqueness of solutions for (1.1)-(1.2) is a non-trivial problem since functions of the form $u \mapsto u(\sigma(u))$ are (in general) nonlinear and not Lipschitz on spaces of continuous functions (which has a direct implication on existence and uniqueness of solutions defined on the whole interval $[0, a]$). To prove our results we introduce a new approach, which is interesting in itself and can be applied to study other types of abstract and applicable problems. Noting that

$$\| u(\sigma(u)) - v(\sigma(v)) \| \leq (1 + [v]_{C_{Lip}([0,a],X)}[\sigma]_{C_{Lip}(C([0,a];X);[0,a])} \| u - v \|_{C([0,a];X)},$$

when the involved functions are Lipschitz, we prove our results working on spaces of Lipschtiz functions (which is a hard problem in the framework of semigroup theory).

In Theorem 2.1 we establish existence and uniqueness of mild and strict solutions assuming that $H(\cdot)$ and $F(\cdot)$ are bounded and a condition guaranteeing that $T(\cdot)H(\sigma(u), u) \in C_{Lip}([0, a]; X)$. In Theorem 2.2 we prove the existence and uniqueness of a mild solution in $C_{Lip,1}([0, a]; X)$ assuming that $H(\cdot)$ and $F(\cdot)$ are bounded. The case where $H(\cdot)$ and $F(\cdot)$ are unbounded can be studied easily

2000 *Mathematics Subject Classification.* 34A10, 34A99, 47H10, 34K30, 47D06 .
Key words and phrases. Non-local conditions, mild solution, strict solutions, analytic semigroup.
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