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Mané Harutyunyan, Bernd Simeon

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Mané Harutyunyan*, Bernd Simeon

Technische Universität Kaiserslautern, Paul-Ehrlich-Straße 31, 67663 Kaiserslautern

Abstract

This paper deals with the analysis of a coupled problem arising from linear magneto-elastostatics. The model, which can be derived by an energy principle, gives valuable insight into the coupling mechanism and features a saddle point structure with the elastic displacement and magnetic scalar potential as independent variables. As main results, the existence and uniqueness of the solution are proven for the continuous and discrete cases and special properties of the corresponding bilinear forms are shown. In particular, the coupled magneto-elastic bilinear form satisfies an inf-sup condition for a certain class of magnetostrictive materials, that is essential for the stability of the problem.

Keywords:
Magneto-elastic coupling, magnetostriction, mixed problem, mathematical modeling
2000 MSC: 35J57, 35Q61, 35Q74, 35M32

1. The coupled problem

We consider materials that are *magnetostrictive*, i.e. they change their shape or dimensions as a result of magnetization. Due to the mutual coupling of mechanical and magnetic fields, these materials have wide application areas, e.g. as variable-stiffness devices, sensors and actuators in mechanical systems or artificial muscles.
Basic references providing a theoretical background on magneto-elastic coupling are, among others, Brown [1], Pao [2], Truesdell and Taupin [3] and Hutter and Van den Veen [4], while Engdahl(ed.)[5] gives a thorough insight into the modeling and application of magnetostrictive materials. In view of the coupled magneto-elastic model we present in this paper, an extensive treatment of the theory and applications of mixed finite element methods and saddle point problems can be found in the book of Boffi, Brezzi and Fortin [6].
The material is assumed to be homogeneous at the macroscopic level and forms a body whose undeformed state we denote by the domain $\Omega \in \mathbb{R}^3$. Let Γ_D and $\Gamma_\text{mag,D}$ describe the elastic and magnetic Dirichlet boundaries of the material, respectively, while Γ_N and $\Gamma_\text{mag,N}$ denote the corresponding Neumann boundaries. We aim at examining its behavior due to the influence of external magnetic and mechanical fields. Our model is based on a linearized theory, which means that only small strains and small deviations from the initial magnetized state are considered and the magneto-elastic coupling occurs in the material itself. The deformation of the body is characterized by the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and the strain $\boldsymbol{\epsilon} : \Omega \to \mathbb{R}^{3\times 3}$, $\boldsymbol{\epsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$, while the magnetic influence is described by the magnetic field $\boldsymbol{H} : \Omega \to \mathbb{R}^3$. The existence of a magnetic scalar potential $\Psi : \Omega \to \mathbb{R}$ with $\boldsymbol{H} = -\nabla\Psi$ is justified by assuming a static magnetic field that is generated by permanent magnets rather than a current-carrying coil, reducing Maxwell's equations to $\operatorname{curl} \boldsymbol{H} = \boldsymbol{0}$ (see e.g. [7]). Furthermore, $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{3\times 3}$ and $\boldsymbol{B} : \Omega \to \mathbb{R}^3$ describe the stress and the magnetic flux density. The Cauchy stress $\boldsymbol{\sigma}$ and the strain $\boldsymbol{\epsilon}$ are second order tensors.
Although magnetostrictive materials show a non-linear behavior in general, they can be reasonably characterized in case of small strains and magnetizations by means of the linearized constitutive equations of piezomagnetic materials [5, 8], which read

$$\boldsymbol{\sigma}(\boldsymbol{u}, \Psi) = \boldsymbol{C^H} : \boldsymbol{\epsilon}(\boldsymbol{u}) + \boldsymbol{e} \cdot \nabla\Psi, \quad \boldsymbol{B}(\boldsymbol{u}, \Psi) = \boldsymbol{e^T} : \boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{\mu}^\epsilon \nabla\Psi. \tag{1}$$

Here $\boldsymbol{C}^H \in \mathbb{R}^{3\times3\times3\times3}$ denotes the linear elastic stiffness tensor (of order four) for a constant magnetic field and $\boldsymbol{e} \in \mathbb{R}^{3\times3\times3}$ the magneto-elastic coupling tensor (of order three). Furthermore, $\boldsymbol{\mu}^\epsilon \in \mathbb{R}^{3\times3}$ is the symmetric (second order) magnetic permeability tensor for constant strain.
In the above system of equations, we use the inner product of tensors of order 1 and higher,

*Corresponding author
Email addresses: harutyun@mathematik.uni-kl.de (Mané Harutyunyan), simeon@mathematik.uni-kl.de (Bernd Simeon)