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Several sufficient conditions on the absence of global solutions of the fifth-order KdV equations with $L^1(\mathbb{R})$ initial data

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ABSTRACT

This paper studies the absence of the global solutions to the fifth-order KdV equations in the form,

$$\partial_t u + \partial_x^5 u = b_1 u \partial_x u + c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+.$$

As the initial data $u(0,x) \in L^1(\mathbb{R})$, the coefficients $b_1 > -c_1$ and $c_1 = c_2 < 0$, we use the method of nonlinear capacity, developed by Galaktionov, Pokhozhaev and Mitidieri, and obtain several sufficient conditions of nonexistence of global solutions.

1. Introduction

In this paper, we study the singular solutions of the fifth-order KdV equations in the form

$$\partial_t u - \partial_x^5 u + b_1 u \partial_x u + c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u = 0, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+. \tag{1.1}$$

where b_1, c_1, c_2 are real constants, the initial data $u_0(x) = u(0,x) \in L^1(\mathbb{R})$ is given, and the function $u(t,x)$ is unknown.

Eq. (1.1) has been derived to model many physical phenomena such as gravity–capillary waves on a shallow layer and magneto-sound propagation in plasmas as well as the interaction effects between long and short waves [1,2].

A special case for the fifth-order KdV equation (1.1) is the Kawahara equation, i.e., $c_1 = c_2 = 0$ in (1.1). The Cauchy problems of the Kawahara equation as initial data $u_0 \in H^s(\mathbb{R})$ has received much attention (see [3–9] and references therein). The latest work is obtained by Kato [9], who showed that it is globally well-posed as $u_0 \in H^s(\mathbb{R})$ with $s \geq -\frac{38}{21}$, and ill-posed as $u_0 \in H^s(\mathbb{R})$ with $s < -2$ in the sense that the data to solution mapping $H^s(\mathbb{R}) \to C([0,T]; H^s(\mathbb{R}))$ is not Lipschitz continuous.

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But the study of the fifth-order KdV equation (1.1) with $c_2 \neq 0$ is much less advanced (cf. [10–16]). This is due to the fact that the dispersive smoothing effects of the corresponding linear equation

$$\partial_t u + \partial_x^5 u = 0 \tag{1.2}$$

is not strong enough to control the third order nonlinear term $u\partial_x^3 u$. Most recently, Kenig–Pilod [11], and Guo et al. [10] showed independently that the IVP (1.1) is globally well-posed in the space $H^s(\mathbb{R})$ for any $s \geq 2$ as $c_1 = 2c_2$ by using the short-time $X^{s,b}$ structure method and a priori estimates method. Also the results in [15] need $\xi^{-\frac{1}{4}}\widehat{u}(0,\xi) \in L^2(\mathbb{R})$.

However, as $c_1 = c_2$ and $u_0 \in L^1(\mathbb{R})$ whether (1.1) obtains a unique global solution or blow-up solution was not yet known. With this motivation, we will use the method of nonlinear capacity (see [17–19]), to analyze the crash solutions of (1.1), i.e., when the corresponding integral of the form $\int_\Omega u(t,x)\phi(t,x)dx$ tends to infinity as $t \to T$ for some $\Omega \subset \mathbb{R}$ and some appropriate weight function $\phi(t,x)$.

The remaining part of this paper is organized as follows. In Section 2, we present some notations and two results in this paper. In Section 3, we give the exact proof to the two results in this paper.

2. Main results

Before presenting our main results in this paper, we need to introduce a special function $\phi_L(x)$ as a preparation.

Suppose $u(x,t)$ is a solution to (1.1), let

$$J_L(t) = \int_{-L}^{L} u(t,x)\phi_L(x)dx,$$

where $\phi_L(\cdot) \in C^5([-L,L])$ with the following properties,

$$\phi_L'(x) = \frac{d\phi_L(x)}{dx} \geq 0, \qquad \text{as } x \in [-L,L], \tag{2.1}$$

$$\phi_L'''(x) = \frac{d^3\phi_L(x)}{(dx)^3} \geq 0, \qquad \text{as } x \in [-L,L], \tag{2.2}$$

$$\phi_L^{(k)}(x)|_{x=L} = \frac{d^k\phi_L(x)}{(dx)^k}|_{x=L} = 0, \qquad k = 0,1,2,3,4. \tag{2.3}$$

$$0 < \int_{-L}^{L} \frac{(\partial_x^5\phi_L)^2}{\partial_x\phi_L}dx + \int_{-L}^{L} \frac{(\phi_L)^2}{\partial_x\phi_L}dx + \int_{-L}^{L} \frac{(\phi_L)^2}{\partial_x^3\phi_L}dx \leq A < \infty, \tag{2.4}$$

where the constant A is independent of L.

Remark 1: In fact, we can find many smoothing functions $\phi_L(\cdot)$ satisfying (2.1)–(2.4), for example,

$$\phi_L(x) = \frac{(x-L)^j}{L^m}, \qquad j > 9 \text{ and } m \geq j+4.$$

Now, multiplying both sides of (1.1) by ϕ_L, integrating over $[-L,L]$ as well as applying the integration by parts formula for several times, we arrive at

$$\begin{aligned} &\frac{d}{dt}J_L(t) + \int_{-L}^{L} u\partial_x^5\phi_L dx - \frac{b_1}{2}\int_{-L}^{L} u^2\partial_x\phi_L dx \\ &+(c_2-c_1)\int_{-L}^{L} u\partial_x^3 u\phi_L dx + c_1\int_{-L}^{L}(\partial_x u)^2\partial_x\phi_L dx - \frac{c_1}{2}\int_{-L}^{L} u^2\partial_x^3\phi_L dx \\ &+B_{u,\phi}(x,t)|_{x=-L} \\ =\;& 0, \end{aligned} \tag{2.5}$$