Analyticity of the inhomogeneous incompressible Navier–Stokes equations

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ABSTRACT. In this paper, we obtain analyticity of the inhomogeneous Navier-Stokes equations. The main idea is to use the exponential operator $e^{\phi(t)|D|}$, where $\phi(t)=\delta-\theta(t)$, $\delta>0$ is the analyticity radius of (ρ_0-1,u_0), and $|D|$ is the differential operator whose symbol is given by $\|\xi\|_{l^1}$. We will show that for sufficiently small initial data, solutions are analytic globally in time in critical Besov spaces.

1. INTRODUCTION

In this paper, we study the inhomogeneous incompressible Navier-Stokes equations governing the time evolution of the density ρ, the velocity u and the pressure π in $\mathbb{R}^3$:

$$\rho_t+u\cdot\nabla\rho=0, \tag{1.1a}$$

$$\rho\left(u_t+u\cdot\nabla u\right)+\nabla\pi-\Delta u=0,\quad \text{div}u=0. \tag{1.1b}$$

For the mathematical background of this model, see [9]. When ρ_0 is away from zero, we take the density ρ which is a small perturbation of the constant density, say 1. In this case, we take $\eta=1-\frac{1}{\rho}$ and rewrite (1.1) in terms of (u,η,π) as

$$\eta_t+u\cdot\nabla\eta=0, \tag{1.2a}$$

$$u_t-\Delta u+\nabla\pi=-u\cdot\nabla u-\eta\nabla\pi+\eta\Delta u,\quad \text{div}u=0. \tag{1.2b}$$

We note that (1.2) satisfies the scaling invariant property: if (u,η,π) solves (1.2), so does

$$u_\lambda(t,x)=\lambda u\left(\lambda^2t,\lambda x\right),\quad \eta_\lambda(t,x)=\eta\left(\lambda^2t,\lambda x\right),\quad \pi_\lambda(t,x)=\lambda^2\pi\left(\lambda^2t,\lambda x\right).$$

In this paper, we choose the following scaling invariant Besov spaces for initial data

$$u_0\in\dot{B}^{\frac{3}{p}-1}_{p,1},\quad \eta_0\in\dot{B}^{\frac{3}{p}}_{p,1}. \tag{1.3}$$

We require that η stay in a Banach algebra $\dot{B}^{\frac{3}{p}}_{p,1}$ to deal with the product of η with u and p. Let

$$\begin{aligned}\mathcal{I}_0&=\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+\|\eta_0\|_{\dot{B}^{\frac{3}{p}}_{p,1}},\\ \|(u,\eta,\pi)\|_{\mathcal{E}_T}&=\|u\|_{\widetilde{L^\infty_T}\dot{B}^{\frac{3}{p}-1}_{p,1}}+\|u\|_{L^1_T\dot{B}^{\frac{3}{p}+1}_{p,1}}+\|\eta\|_{L^\infty_T\dot{B}^{\frac{3}{p}}_{p,1}}+\|\pi\|_{L^1_T\dot{B}^{\frac{3}{p}}_{p,1}},\end{aligned} \tag{1.4}$$

where the regularity of π is determined by the following elliptic PDE:

$$-\Delta\pi=\text{div}\left(u\cdot\nabla u+\eta\nabla\pi+\eta\Delta u\right). \tag{1.5}$$

We here restrict take $p<3$ for the well-posedness of (1.2): when $p\geq 3$, $\frac{3}{p}-1$ is nonpositive and so it is impossible to bound $\eta\nabla\pi$ and $\eta\Delta u$ in the right-hand side of (1.2b) in $\mathcal{E}_T$.

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