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Well-posedness of a Debye type system endowed with a full wave equation

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ABSTRACT

We prove well-posedness for a transport-diffusion problem coupled with a wave equation for the potential. We assume that the initial data are small. A bilinear form in the spirit of Kato's proof for the Navier–Stokes equations is used, coupled with suitable estimates in Chemin–Lerner spaces. In the one dimensional case, we get well-posedness for arbitrarily large initial data by using Gagliardo–Nirenberg inequalities.

1. Introduction

Transport-diffusion equations have a vast phenomenology and have been widely studied. See, among others, [1–4] in the case of the semi-conductor theory, and [5] in the case of Fokker–Planck equations. The goal of this note is to prove existence and uniqueness of the solution for a modified semi-conductor equation.

In order to simplify the presentation, we restrict to the case of a single electrical charge. The novelty of our equations is that we replace the Poisson equation on the potential by a wave equation. This is a quite natural change, since the electric charge itself depends on the time. From a mathematical point of view, switching from a Poisson equation to a wave equation roughly amounts to the loss of one derivative in the estimates on the potential. Moreover, it seems that one is bound to work in L_t^p spaces with $1 \leq p \leq 2$ due to the usual Strichartz estimates.

In this paper, we prove the existence of a mild solution in Chemin–Lerner spaces $\tilde{L}^1(0, T, \dot{H}^{n/2-1}(\mathbb{R}^n))$. We first restrict to the case of small initial data ($n \geq 2$), and use a variant of the Picard fixed point theorem as in the proof of Kato's and Chemin's theorems for the Navier–Stokes (and related) equations. See [6–8] and also [9,10]. In particular, we work in homogeneous Sobolev spaces in order to get T-independent estimates

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for the heat equation. Note also that our bilinear form depends on a nonlocal term, given as the solution of the wave equation on the potential.

In the case $n = 1$, well posedness is established for arbitrary large initial data (Section 4). Local well posedness is obtained as in Section 3. The global existence is proved by combining the usual L^1 estimate with a Gagliardo–Nirenberg inequality, in the spirit of [2].

2. Equations and preliminary results

We begin with some notations. In this section $n \geq 2$, $T > 0$, and $s < n/2$ are given. The homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ are often denoted by $\dot{H}^s$. For $p \geq 1$, we also use the Chemin–Lerner spaces $\tilde{L}^p(0, T, \dot{H}^s(\mathbb{R}^n)) = \tilde{L}^p(0, T, \dot{B}^s_{2,2}(\mathbb{R}^n))$, or simply $\tilde{L}^p_T(\dot{H}^s)$. Recall that a distribution $f \in \mathscr{S}'\big(]0, T[\times \mathbb{R}^n\big)$ belongs to the space $\tilde{L}^p_T(\dot{H}^s)$ iff $\dot{S}_j f \to 0$ in $\mathscr{S}'$ for $j \to -\infty$, and $\|f\|_{\tilde{L}^p_T(\dot{H}^s)} := \|(2^{js}\|\dot{\Delta}_j f\|_{L^p_T(L^2)})_{j\in\mathbb{Z}}\|_{l^2(\mathbb{Z})} < \infty$. Here, $\dot{S}_j f$ and $\dot{\Delta}_j f$ are respectively the low frequency cut-off and the homogeneous dyadic block defined by the usual Paley–Littlewood decomposition. See [10] p.98 for details. Last, we write ∇ for the (spatial) gradient, div for the divergence and $\Delta = div\nabla$.

We now give the equations we are dealing with. Set $s = n/2 - 1$. Consider the Cauchy problem on the scalar valued functions u and V defined on $\mathbb{R}_+ \times \mathbb{R}^n_x$

$$\partial_t u - \Delta u = div(u\nabla V) \tag{1}$$

$$\partial_{tt} V - \Delta V = u \tag{2}$$

$$u(0) = u_0 \tag{3}$$

$$V(0) = V_0, V_t(0) = V_1 \tag{4}$$

For $u_0 \in \dot{H}^s$, $(\nabla V_0, V_1) \in \dot{H}^s \times \dot{H}^s$ and $u \in \tilde{L}^1_T(\dot{H}^s)$ given, we denote by $S(u, V_0, V_1) \in C^0\big(0, T, \mathscr{S}'(\mathbb{R}^n)\big)$ the unique solution of the wave equation (2), (4). With these notations, the system (1)–(4) is interpreted as the following problem (P):

find $u \in \tilde{L}^1_T(\dot{H}^s)$ such that

$$\partial_t u - \Delta u = div(u\nabla S(u, V_0, V_1)) \tag{5}$$

$$u(0) = u_0 \tag{6}$$

For future reference, we recall a standard result on the heat equation (see [10] p.157)

Proposition 2.1. *Let $T > 0$, $\sigma \in \mathbb{R}^n$ and $1 \leq p \leq \infty$. Assume that $u_0 \in \dot{H}^\sigma$ and $f \in \tilde{L}^p_T(\dot{H}^{\sigma-2+\frac{2}{p}})$. Then the problem*

$$\partial_t u - \Delta u = f \tag{7}$$

$$u(0) = u_0 \tag{8}$$

admits a unique solution $u \in \tilde{L}^p_T(\dot{H}^{\sigma+\frac{2}{p}}) \cap \tilde{L}^\infty_T(\dot{H}^\sigma)$ and there exists $C > 0$ independent of T such that, for any $q \in [p, \infty]$

$$\|u\|_{\tilde{L}^q_T(\dot{H}^{\sigma+\frac{2}{q}})} \leq C\big(\|f\|_{\tilde{L}^p_T(\dot{H}^{\sigma-2+\frac{2}{p}})} + \|u_0\|_{\dot{H}^\sigma}\big) \tag{9}$$

Moreover, for $f = 0$, we have $u \in C^0([0, T], \dot{H}^\sigma) \hookrightarrow L^1([0, T], \dot{H}^\sigma)$.

The same statements hold true in nonhomogeneous Sobolev spaces with a constant $C = C_T$ depending on T.