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Hypersingular nonlinear boundary-value problems with a small parameter

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A B S T R A C T

Some hypersingular nonlinear boundary-value problems with a small parameter ε at the highest derivative are described. These problems essentially (qualitatively and quantitatively) differ from the usual linear and quasilinear singularly perturbed boundary-value problems and have the following unusual properties:

(i) in hypersingular boundary-value problems, super thin boundary layers arise, and the derivative at the boundary layer can have very large values of the order of $e^{1/\varepsilon}$ and more (in standard problems with boundary layers, the derivative at the boundary has the order of ε^{-1} or less);

(ii) in hypersingular boundary-value problems, the position of the boundary layer is determined by the values of the unknown function at the boundaries (in standard problems with boundary layers, the position of the boundary layer is determined by coefficients of the given equation, and the values of the unknown function at the boundaries do not play a role here);

(iii) hypersingular boundary-value problems do not admit a direct application of the method of matched asymptotic expansions (without a preliminary nonlinear point transformation of the equation under consideration).

Examples of hypersingular nonlinear boundary-value problems with ODEs and PDEs are given and their exact solutions are obtained.

1. Introduction

Singularly perturbed boundary-value problems with a small parameter at the highest derivative are often encountered in hydro- and aerodynamics, theory of mass and heat transfer, theory of elasticity, nonlinear mechanics and other applications. An important qualitative feature of singular boundary-value problems is that for the zero value of a small parameter the order of the differential equation under consideration

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decreases and some parts of the boundary conditions cannot be satisfied. Various problems and solution methods for ODEs and PDEs with a small parameter at the highest derivative are described, for example, in [1–13].

For singularly perturbed quasilinear boundary-value problems of the form

$$\varepsilon y''_{xx} + p y'_x + q(x,y) = 0 \quad (0 < x < 1); \quad y(0) = \alpha, \quad y(1) = \beta \tag{1}$$

with a small parameter ε, the position of the boundary layer is determined by the sign of the coefficient p. For $p > 0$, the boundary layer is formed at the left boundary near the point $x = 0$, for $p < 0$, at the right boundary near the point $x = 1$. An analogous situation holds for more complex singularly perturbed quasilinear boundary-value problems of the form (1) with $p = p(x)$.

We note that for $p > 0$ and $\varepsilon \to 0$ the derivative of the solution of the problem (1) takes large values proportional to ε^{-1} on the left boundary.

In this article we will consider hypersingular boundary-value problems with a small parameter, the solutions of which qualitatively and quantitatively differ from the solutions of the problems (1).

2. Hypersingular boundary-value problems for ordinary differential equations with a small parameter

2.1. Example of hypersingular boundary-value problem, exact solution, qualitative features

Problem 1. Consider the nonlinear boundary-value problem

$$\varepsilon y''_{xx} + p(y'_x)^2 + q y'_x = 0; \tag{2}$$

$$y(0) = \alpha, \quad y(1) = \beta. \tag{3}$$

In what follows, we assume that $p > 0$, $q > 0$ and $\varepsilon > 0$ is a small parameter.

Eq. (2) admits an exact linearization by means of the transformation $y = (\varepsilon/p)\ln u$ (see also Section 2.2). Its general solution is determined by the formula

$$y = \frac{\varepsilon}{p}\ln(C_1 + C_2 e^{-qx/\varepsilon}), \tag{4}$$

where C_1 and C_2 are arbitrary constants.

The exact solution of the boundary-value problem (2)–(3) has the form

$$y = \frac{\varepsilon}{p}\ln\left[\frac{(e^{\alpha p/\varepsilon} - e^{\beta p/\varepsilon})e^{-qx/\varepsilon} + e^{\beta p/\varepsilon} - e^{(\alpha p - q)/\varepsilon}}{1 - e^{-q/\varepsilon}}\right]. \tag{5}$$

Let us find the derivatives on the left and right boundaries:

$$y'_x|_{x=0} = \frac{q}{p}\,\frac{e^{(\beta-\alpha)p/\varepsilon} - 1}{1 - e^{-q/\varepsilon}}, \qquad y'_x|_{x=1} = \frac{q}{p}\,\frac{1 - e^{(\alpha-\beta)p/\varepsilon}}{e^{q/\varepsilon} - 1}. \tag{6}$$

The analysis shows that, depending on the values of the parameters α and β (as $\varepsilon \to 0$), the following three qualitatively different situations are possible:

$$\begin{array}{ll} \text{(i) } \beta > \alpha & \text{(there is a boundary layer on the left, near } x = 0\text{);} \\ \text{(ii) } \beta < \alpha - (q/p) & \text{(there is a boundary layer on the right, near } x = 1\text{);} \\ \text{(iii) } \alpha - (q/p) \le \beta \le \alpha & \text{(there is no boundary layer).} \end{array} \tag{7}$$