Applied Mathematics Letters 80 (2018) 41–47

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

On the missing eigenvalue problem for Dirac operators

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ARTICLE INFO

Article history:
Received 5 November 2017
Received in revised form 4 January 2018
Accepted 4 January 2018
Available online 11 January 2018

MSC:
34A55
34L05
34L40
45C05

Keywords:
Inverse spectral problem
Dirac operator
Missing eigenvalue problem
Potential

ABSTRACT

The inverse spectral problem for Dirac operators is studied. By using the result on the Weyl m-function, we show that the Hochstadt–Lieberman-type and Borg-type theorem for Dirac operators except for one arbitrary eigenvalue hold.

1. Introduction

Consider the following Dirac operator $L := L(V_1, V_2, \alpha, \beta)$ defined by

$$\begin{cases} u_2' + V_1(x)u_1 = \lambda u_1, \\ -u_1' + V_2(x)u_2 = \lambda u_2, \end{cases} \quad x \in (0, \pi), \tag{1.1}$$

with boundary conditions

$$u_1(0, \lambda)\sin\alpha + u_2(0, \lambda)\cos\alpha = 0, \tag{1.2}$$

$$u_1(\pi, \lambda)\sin\beta + u_2(\pi, \lambda)\cos\beta = 0, \tag{1.3}$$

where $0 \le \alpha, \beta < \pi$, λ is the spectral parameter, $V_k(x), k = 1, 2$, are absolutely continuous real-valued functions. In the case where $V_1(x) = V(x) + m$ and $V_2(x) = V(x) - m$, $V(x)$ is a potential function and

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https://doi.org/10.1016/j.aml.2018.01.004

$m > 0$ is the mass of a particle, (1.1) is called a one-dimensional stationary Dirac system in relativistic quantum theory (see [1,2]). Direct and inverse problems for (1.1)–(1.3) have been studied in [1–5] and other works.

The inverse spectral problem consists in recovering this operator from its spectral characteristics. Inverse problems have many applications in mathematics, natural sciences and engineering (see [2,6] and other works). Inverse problems have been studied fairly completely for Sturm–Liouville operators (see, [1,2,6–12] and the references therein). In particular, using the Weyl-m function techniques, Gesztesy and Simon [8] solved an inverse problem involving fractions of the eigenvalues of the Sturm–Liouville operator on a finite interval and knowledge of the potential over a corresponding fraction of the interval, which is a generalization of the Hochstadt–Lieberman theorem [9]. In 2009, Wei and Xu [12] solved an open problem of missing one eigenvalue presented by Gesztesy and Simon [8] and showed that if the potential is prescribed on the half interval and coefficients of boundary conditions are known a priori, then all eigenvalues of missing one eigenvalue is sufficient to determine the potential on the whole interval. The necessary and sufficient conditions on the missing eigenvalue problem for the Hochstadt–Lieberman theorem for the Sturm–Liouville operator was considered in [10]. The results on missing eigenvalue problems for differential pencils were found in [13]. In this paper we study the missing eigenvalue problems for Dirac operator.

2. Preliminaries

Let $(u_1(x,\lambda), u_2(x,\lambda))^T$ be the solution of the system of (1.1) with initial conditions $u_1(0,\lambda) = \cos\alpha$ and $u_2(0,\lambda) = -\sin\alpha$. By the method of successive approximations [2], we have

$$u_1(x,\lambda) = \cos(\lambda x - \eta(x) - \alpha) + \frac{U_1(x,\lambda)}{\lambda} + o\left(\frac{e^{\tau x}}{\lambda}\right), \tag{2.1}$$

$$u_2(x,\lambda) = \sin(\lambda x - \eta(x) - \alpha) + \frac{U_2(x,\lambda)}{\lambda} + o\left(\frac{e^{\tau x}}{\lambda}\right) \tag{2.2}$$

for $|\lambda| \to \infty$, uniformly in $x \in [0,\pi]$, where $\tau = |\mathrm{Im}\lambda|$, $\eta(x) = \frac{1}{2}\int_0^x [V_1(t) + V_2(t)]\mathrm{d}t$ and

$$\begin{aligned} U_1(x,\lambda) = {} & \frac{v_{1,2}(0)}{4}\cos(\lambda x - \eta(x) + \alpha) - \frac{v_{1,2}(x)}{4}\cos(\lambda x - \eta(x) - \alpha) \\ & + \frac{1}{8}\int_0^x v_{1,2}^2(t)\mathrm{d}t \cdot \sin(\lambda x - \eta(x) - \alpha), \\ U_2(x,\lambda) = {} & \frac{v_{1,2}(x)}{4}\sin(\lambda x - \eta(x) - \alpha) + \frac{v_{1,2}(0)}{4}\sin(\lambda x - \eta(x) + \alpha) \\ & - \frac{1}{8}\int_0^x v_{1,2}^2(t)\mathrm{d}t \cdot \cos(\lambda x - \eta(x) - \alpha), \end{aligned}$$

where $v_{1,2}(x) = V_1(x) - V_2(x)$. The characteristic function $\Delta(\lambda)$ of (1.1)–(1.3) is defined by the following relation:

$$\Delta(\lambda) := u_1(\pi,\lambda)\sin\beta + u_2(\pi,\lambda)\cos\beta, \tag{2.3}$$

and all zeros $\lambda_n, n \in \mathbb{Z}$ of $\Delta(\lambda)$ coincide with the eigenvalues of L. Denote $\sigma(L) := \{\lambda_n : n \in \mathbb{Z}\}$. Then all eigenvalues λ_n are real and simple, and satisfy the asymptotic formula (see [2]):

$$\lambda_n = n + \frac{\omega_1}{\pi} + \frac{\omega_2}{n} + o\left(\frac{1}{n}\right), \tag{2.4}$$

where

$$\omega_1 = \alpha - \beta + \eta(\pi), \quad \omega_2 = \frac{1}{4\pi\cos^2\omega_1}\left[v_{1,2}(\pi)\sin(2\beta) - v_{1,2}(0)\sin(2\alpha) + \frac{1}{2}\int_0^\pi v_{1,2}^2(t)\mathrm{d}t\right].$$