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Stability of a nonlocal delayed reaction–diffusion equation with a non-monotone bistable nonlinearity

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ABSTRACT

The present work is devoted to the stability and attractivity analysis of a nonlocal delayed reaction–diffusion equation (DRDE) with a non-monotone bistable nonlinearity that describes the population dynamics for a two-stage species with Allee effect. By the idea of relating the dynamics of the nonlinear term to the DRDE and some stability results for the monostable case, we describe some basin of attractions for the DRDE. Additionally, existence of heteroclinic orbits and periodic oscillations are also obtained. Numerical simulations are also given at last to verify our theoretical results.

1. Introduction

When modeling the growth of mature population of a two-stage species (juvenile and adult, with a fixed maturation time τ) whose both mature individuals and immature individuals diffuse, one faces a nonlocal delayed reaction–diffusion equation of the form

$$\frac{\partial u}{\partial t}(t,x) = D_m \Delta u(t,x) - d_m u(t,x) + \epsilon \int_{\Omega} \Gamma(\alpha,x,y) b(u(t-\tau,y))\mathrm{d}y,\ (t,x)\in(0,\infty)\times\Omega. \tag{1.1}$$

Here $u(t,x)$ represents the total mature population at time t and location x. D_m and d_m are the diffusion rate and death rate for the mature population, τ is the maturation time for the species, the other two indirect parameters ε and α are defined by $\varepsilon = e^{-\int_0^{\tau} d_I(a)\mathrm{d}a}$ and $\alpha = \int_0^{\tau} D_I(a)\mathrm{d}a$, where $d_I(a)$, $D_I(a)$, $a \in [0,\tau]$ are the age dependent death rate and diffusion rate of the immature population of the species, $b(u)$ is the birth function. The kernel $\Gamma(\alpha,x,y)$ parameterized by α accounts for the probability that an individual born at location y can survive the immature period $[0,\tau]$ and has moved to location x when becoming mature

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(τ time units after birth). When mature individuals of the species diffuse but their immature individuals do not, such as birds, by letting $\alpha \to 0$, model (1.1) is reduced to the following spatially local model

$$\frac{\partial u(t,x)}{\partial t} = D_m \Delta u(t,x) - d_m u(t,x) + \epsilon b(u(t-\tau,x)),\ (t,x) \in (0,\infty)\times\Omega. \tag{1.2}$$

Depending on the situations of the domain Ω, different problems may arise. In the case $\Omega = \mathbb{R}$, So et al. [1] obtained that $\Gamma(\alpha,x,y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{(x-y)^2}{4\alpha}}$. In this case, traveling wave solutions, which may explain spatial invasion of species are of paramount significance and have been successfully investigated by many researchers for both models (1.1) and (1.2). See, for example, [1–5] and the references therein. Recently, the global dynamics of model (1.1) on $\mathbb{R}$ and $\mathbb{R}_+ = [0,\infty)$ have also been investigated by Yi et al. in [6] and Yi and Zou in [7].

When the spatial domain Ω is bounded, the kernel $\Gamma(\alpha,x,y)$ is related to the semigroup $S(t)$ generated by the operator $-\Delta$ under corresponding boundary conditions on $\partial\Omega$ in the sense that $S(\alpha)[\phi](x) = \int_\Omega \Gamma(\alpha,x,y)\phi(y)\mathrm{d}y$. Generally speaking, explicit forms of $\Gamma(\alpha,x,y)$ can only be obtained for some special cases, see [8]. In the case a Dirichlet boundary value condition(DBVC) is posed, existence, uniqueness and attractiveness of a positive steady state and threshold dynamics are of main concerns and have been explored extensively and intensively by many authors adopting various methods (see, e.g., [9–11]). When a Neumann boundary value condition (NBVC) is imposed, Yi and Zou [12] obtained threshold dynamics for model (1.2). Their methods were extended to describe the global dynamics of (1.1) in [13]. Zhao [14] has recently established the global attractiveness of the positive steady state of (1.1) by adopting a fluctuation method. In the very recent work, Su and Zou [15] established the existence of periodic solutions by investigating the Hopf bifurcations of (1.1) when $\Omega = [0,\pi]$.

However, all the above mentioned results are obtained in the case that the nonlinearity b is monostable, namely, model (1.1) has only one positive equilibrium. To the best of the author's knowledge, little attention has been paid to dynamics of neither model (1.1) nor (1.2) when the nonlinearity b is bistable. Therefore, we investigate the dynamics of model (1.1) under a NBVC with a bistable nonlinear term. Our work is mainly inspired by the recent works [12] and [16]. In [16], the authors explored the stability and basin of attraction for a class of delay differential equations by combining the idea of relating the dynamics of a map to the dynamics of a delay differential equation and invariance arguments for the solution semiflow. Similarly, we obtain our results by delicately analyzing the dynamics of the nonlinearity b, combined with some results established in [12] for monostable case. By the definition of $\Gamma(\alpha,x,y)$, one can see that $S(\alpha)[a] = \int_\Omega \Gamma(\alpha,x,y)a\mathrm{d}y = a$ for any constant a under NBVC and hence the results established in [12] for model (1.2) also hold for model (1.1)(details can be found in [13]), which enables us to directly employ the results in [12].

The remaining of this paper is organized as follows. In Section 2, we first give some notations and then explore the equilibria and properties of the nonlinear term. Main results are presented in Section 3, where we obtain the attractivity of the equilibria on certain intervals and the existence of two types of heteroclinic orbits: orbit from one equilibrium to another one, and orbit from one equilibrium to a periodic orbit oscillating around the largest positive equilibrium, implying a Hopf bifurcation.

2. Preliminary results

Let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}_+$ be the sets of all reals, positive reals and positive integers respectively, and $\tau \in \mathbb{R}^+$ be a given positive constant. Suppose that Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ is the Laplacian operator on Ω and $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative on $\partial\Omega$. Let $C = \{\phi \in C(\bar\Omega,\mathbb{R})\}$ and $X = \{\varphi \in C([-1,0]\times\bar\Omega,\mathbb{R})\}$ be equipped with the usual supremum norm $\|\cdot\|$. Let $C_+ = \{\phi \in C : \phi|_{\bar\Omega} \geq 0\}$, $X_+ = \{\varphi \in X : \varphi|_{[-1,0]\times\bar\Omega} \geq 0\}$. For $a \in \mathbb{R}$, $\hat a \in C$ is defined as $\hat a(x) = a$ for all $x \in \Omega$. Similarly, $\hat{\hat a} \in X$ is defined as $\hat{\hat a}(\theta) = \hat a$ for all $\theta \in [-1,0]$. For any $\xi,\eta \in C$, we write $\xi \geq_C \eta$ if $\xi - \eta \in C_+$, $\xi >_C \eta$ if