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On some properties of shock processes in a 'natural' scale

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ABSTRACT

We consider shocks modeling in a 'natural' scale which is a discrete scale of natural numbers. A system is subject to the shock process and its survival probability and other relevant characteristics are studied in this scale. It turns out that all relations for the probabilities of interest become much easier in the new scale as compared with the conventional chronological time scale. Furthermore, it does not matter what type of the point process of shocks is considered. The shock processes with delays and the analog of a shot-noise process are discussed. Another example of the application of this concept is presented for systems with finite number of components described by signatures.

1. Introduction

Various shock models have been intensively studied in the literature (see, e.g., [10] and references therein) and applied to various reliability topics. For instance, Caballé at al. [4], Montoro-Cazorla et al. [23], Kenzin and Frostig [18], Montoro-Cazorla and Pérez-Ocón [22], van der Weide and Pandey [28] and Ruiz-Castro [5] have considered applications to optimal maintenance modeling. See also [7,9,16,24,27] for general shock-based reliability modeling and analysis. In reliability and safety studies and applications, the most popular model is, probably, the so called, *extreme shock model*, where each shock can result in a system's failure with the specified probability and a system survives it with the complimentary probability. This model can, in principle, account for damage accumulation as well when the probability of failure increases with each survived shock [10]. It should be noted that survival probabilities of systems subject to shock processes for extreme shock model can be obtained explicitly only for the Poisson process of shocks. Even for the renewal processes of shocks, everything becomes more cumbersome and asymptotic or approximate methods should be used for the corresponding calculations. However, probabilistic reasoning can be substantially simplified when we change our usual chronological time scale to the scale defined by the number of shocks occurrences. Of course, this can be made only for systems that allow for this 'transformation'. Thus, in the current paper, instead of a continuous chronological time scale $[0,\infty)$, we will use the discrete one $N=[1,2,\ldots]$ for 'times' to failure of systems.

For illustration of our claim, consider first, a system subject to the orderly (without multiple occurrences) process of shocks in chronological time scale. Assume for simplicity that shocks present the only cause of its failure. In reliability applications, we are usually interested in the probability of survival of a system in k. Denote this probability by k. The simplest model is when an operating system is subject to the homogeneous Poisson process of shocks (with the constant rate λ) and survives each shock with probability q and fails with the complementary probability $p=1-q$. In this case, it is well known that the probability of survival of a system in k is

$$P(t)=\exp\{-p\lambda t\}, \tag{1}$$

whereas for the nonhomogeneous Poisson process (NHPP) with rate $\lambda(t)$ and time dependent $p(t), q(t)$, this formula turns to [1,8]:

$$P(t)=\exp\left\{-\int_0^t p(u)\lambda(u)du\right\}. \tag{2}$$

Obviously, when $\lambda(t)=\lambda$ and $p(t)=p$, (2) turns to (1). Even the simplest analog of the survival probability (1) for a renewal process already cannot be obtained in a similar simple form and, usually, computational methods, bounds or asymptotic methods are used for obtaining the corresponding probability $P(t)$ [17]. The exact relation can be obviously formally written in the form of the

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infinite series as

$$P(t) = \sum_{k=0}^{\infty} q^k (G^{(k)}(t) - G^{(k+1)}(t)) = \sum_{k=1}^{\infty} pq^{k-1}(1 - G^{(k)}(t)), \quad (3)$$

where $G(t)$ is the baseline distribution for a renewal process, $G^{(k)}(t)$ is the k-fold convolution of $G(t)$ with itself, $G^{(1)}(t) \equiv G(t)$, $G^{(0)}(t) \equiv 1$ and $G^{(k)}(t) - G^{(k+1)}(t)$ is the probability of k renewals in $[0, t)$. When $p \to 0$ (3) is characterized by the following asymptotic formula when $p \to 0$ [17]

$$P(t) = \exp\left\{-\frac{pt}{\mu}\right\}(1 + o(1)), \quad (4)$$

where we assume that $\mu = \int_0^\infty \overline{G}(u)du < \infty$.

Eqs. (1)–(4) are derived in a conventional time scale and define the corresponding probabilistic characteristics in a chronological time t. However, there are a lot of settings when we are not actually interested in survival in real time. It is well known that reliability indices can be functions not necessarily of time (as usually) but of other monotone quantities. For instance, the growing crack in a material can be described by its length s, whereas a random variable S is the length of a crack at which a failure occurs. Therefore, the cdf of the time to failure can be parameterized accordingly as $F(s) = P(S \le s)$. Similar with automobiles, where S can represent a random mileage to failure (see, e.g., [15] and [6] on alternative scales). Similar to the above two examples with a continuous alternative scale, we can consider the corresponding number of shocks in a general extreme shock model as a new alternative discrete scale. Thus, a random variable N_S, which is the number of shocks till the system's failure and its discrete distribution $F(k) = P(N_S \le k)$ will be of interest. See also the relevant discussion in [26] and [19].

By a shock we understand an external 'point' event that can result in a system's failure. Usually shocks are described by a random magnitude. However, our description in this paper employs the probability of failure under a shock that is an aggregated characteristic, which already takes into account the shock's magnitude. There are numerous practical examples of shocks effecting operating systems. In electrical systems, the peaks of voltage over a threshold can be considered as shocks. Each shock of this kind can result in a failure of a system, whereas when the fluctuations of voltage are within normal bounds, they are 'harmless'. Hackers attack on computer systems or random missile attacks in warfare can be also considered as shocks, as well as earthquakes, lightning strikes, etc.

There are two main advantages of our "time-free" approach as compared with the conventional case of the time scale

a. All relations for the probabilities of interest become much easier,
b. It does not matter what kind of a shock process is considered: only the number of shocks is relevant!

The importance of the second claim is obvious and it is hard to overestimate it. We will illustrate the first statement with the simplest setting when each shock 'kills' a system with probability p and a system survives with probability q, which describes the extreme shock model. It is obvious that (1) (or (3)) corresponds now to the simplest power function

$$P(k) = q^k, k = 1, 2, \ldots, \quad (5)$$

whereas the discrete distribution of 'time' to failure is given by the following mass and cumulative distribution functions as

$$f(k) = pq^{k-1}, k = 1, 2, \ldots; \quad F(k) = \sum_1^k f(i) = 1 - P(k) = 1 - q^k, \quad (6)$$

respectively. However, as was already emphasized, (5) and (6) do not depend on the type of arriving shock process, whereas (1) is true only for the HPP and (2) holds only for the NHPP.

Thus we see that the probabilities of interest in the new scale are described by the corresponding discrete distributions. Ageing properties can be also formulated in a simpler way in the new scale, as only aging properties of the corresponding discrete distributions matter, whereas, e.g., in (2), the distribution $F(t) = 1 - P(t)$ will be, for instance, IFR when $\lambda(t)p(t)$ is increasing (non-decreasing). Therefore, ageing properties in the latter case obviously also depend on the rate of the arriving shock process. Thus, some properties of discrete distributions and, specifically, the discrete failure rate will play an important role in what follows. As the discrete failure rate is a controversial (in a way) characteristic, in the next section, we would like to discuss some of its properties relevant for our further presentation. Note that the new scale is not a universal one; it should be used in the justified situations when properties in chronological time are not so relevant. For instance, in a continuous case, for the warranties based on mileage, it does not matter how long in time this mileage has been accumulated.

An important example of the alternative discrete scale is when a piece of equipment operates in cycles and the observation is the number of cycles completed before the failure. In this case, we can 'interpret' one cycle of usage as a shock and, therefore, the probability of failure under a shock, e.g., in the model (5) and (6), p can *be equivalently 'interpreted'* as the probability of failure on the corresponding cycle. Thus, our formulation based on shock modeling considered in this paper can be generally applied to such systems. Furthermore, this setting describes a rather broad class of technical systems that are used intermittently [26] and its importance for reliability practice is hard to overestimate. For the meaningful discussion of applications of discrete distributions see [3].

This note is organized as follows. In Section 2 we discuss some properties of the discrete failure rate. In Section 3 some well-known in the chronological scale shock-based setting are 'translated' to the discrete scale. Section 4 deals with shocks affecting the operation of a monotone system of $n$components. Finally, some remarks are given in Section 5.

2. Preliminaries: discrete failure rate

As we are dealing with discrete distributions, we will briefly present now some well-known facts on the discrete failure rate in the manner useful for further presentation and discussion.

We think that there is nothing wrong with the notion of the discrete failure rate except the term itself, as its meaning is slightly different from that in the case of continuous random variables, which sometimes can result in confusion. Moreover, some alterations of the classical definition were suggested to avoid this confusion (see the next section).

For the sake of comparison, denote by T_c the lifetime random variable with absolutely continuous cdf $F(t) = P(T_c \le t)$ and the pdf $f(t) = F'(t)$, the failure rate $\lambda(t) = f(t)/\overline{F}(t)$. When $\Delta(t)$ is sufficiently small, obviously

$$\Pr(t < T_c \le t + \Delta t \mid T_c > t) \approx \lambda(t)\Delta t, \quad (7)$$

whereas $\lambda(t)$ *does not have the meaning of probability itself.* It is well known that the series system of n independent components with