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On an inverse spectral problem for first-order integro-differential operators with discontinuities

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ABSTRACT

A convolution integro-differential operator of the first order with a finite number of discontinuities is considered. Properties of its spectrum are studied and a uniqueness theorem is proven for the inverse problem of recovering the convolution kernel along with the boundary condition from the spectrum.

1. Introduction

The paper deals with an inverse spectral problem for the integro-differential operator that corresponds to the following boundary value problem $L = L(H(x), h, \{a_k\}_{k=1}^N, \{\alpha_k\}_{k=1}^N)$ with discontinuities inside the interval:

$$\ell y(x) := iy'(x) + \int_0^x H(x-t)y(t)\,dt = \lambda y(x), \quad x \in \bigcup_{j=0}^N (a_j, a_{j+1}), \tag{1}$$

$$y(a_j+0) = \alpha_j y(a_j-0), \quad j = \overline{1,N}, \tag{2}$$

$$U(y) := hy(0) - y(\pi) = 0, \tag{3}$$

where $H(x)$ is a complex-valued function, $(\pi - x)H(x) \in L_2(0,\pi)$, $h, \alpha_1, \ldots, \alpha_N \in \mathbb{C} \setminus \{0\}$, $0 = a_0 < a_1 < a_2 < \cdots < a_N < a_{N+1} = \pi$.

Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and

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other branches of natural sciences and engineering. For differential operators with discontinuities inverse problems were studied in [1–11]. The presence of a discontinuity in the mathematical model is connected with discontinuous material properties or with a discontinuity in the corresponding physical process.

For integro-differential and other classes of non-local operators inverse problems are more difficult for investigation, and the classical methods either are not applicable to them or require essential modifications. Various aspects of inverse problems for integro-differential operators without discontinuities were studied in [12–24] and other papers. The presence of the discontinuity conditions (2) essentially complicates studying even the direct problem for L, i.e. the problem, when its spectral properties are to be established. In this paper a special fundamental solution of Eq. (1) is constructed, which allowed to study the properties of the spectrum of L. Namely, the following theorem holds.

Theorem 1. *The problem L has infinitely many eigenvalues λ_k, $k \in \mathbb{Z}$, of the form*

$$\lambda_k = 2k + \omega + \kappa_k, \quad \{\kappa_k\} \in l_2, \tag{4}$$

with account of multiplicities. Moreover, $\omega = (i\pi)^{-1}(\ln(\alpha_1 \dots \alpha_N) - \ln h)$.

Consider the following inverse problem.

Inverse Problem 1. Given the spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$, find the function $H(x)$ and the number h, provided that the values a_j, α_j, $j = \overline{1, N}$, are known a priori.

We will study this inverse problem in the case, when the pair of sets $\{a_k\}_{k=1}^N$ and $\{\alpha_k\}_{k=1}^N$ satisfies the following condition (*Condition $\mathcal{A}$*): $a * b(x) := \int_0^x a(t)b(x-t)\,dt \neq 0$ for $x \in (0, \pi]$, where $a(x) = A_k := \alpha_0\alpha_1 \dots \alpha_k$, $x \in (a_k, a_{k+1})$, $k = \overline{0, N}$, $\alpha_0 = 1$, and $b(x) = A_N/A_{N-k}$, $x \in (\pi - a_{N+1-k}, \pi - a_{N-k})$, $k = \overline{0, N}$. The case, when Condition $\mathcal{A}$ is not fulfilled, requires a separate investigation. Obviously, Condition $\mathcal{A}$ is automatically fulfilled, if all $\alpha_k > 0$. Moreover, it is easy to prove that in the case $N = 1$ Condition $\mathcal{A}$ is equivalent to the condition $\alpha_1 \notin (-\infty, 0]$. In [25] for the case, when $N = 1$ and $a_1 = \pi/2$, the following theorem was proven, which gives uniqueness of solution along with necessary and sufficient conditions for the solvability of Inverse Problem 1 in this special case.

Theorem 2 (*See [25]*). *Let $N = 1$, $a_1 = \pi/2$ and $\alpha_1 \notin (-\infty, 0]$. Then for an arbitrary sequence of complex numbers $\{\lambda_k\}_{k\in\mathbb{Z}}$ of the form (4) there exists a unique (up to values on a set of measure zero) function $H(x)$, $(\pi - x)H(x) \in L_2(0, \pi)$, and a unique number $h \neq 0$, such that $\{\lambda_k\}_{k\in\mathbb{Z}}$ is the spectrum of the corresponding boundary value problem $L(H(x), h, \pi/2, \alpha_1)$.*

The proof of Theorem 2 in [25] is based on the method used in [15], which gives also an algorithm for solving the inverse problem. We note that the general case of $\{a_k\}_{k=1}^N$ is more difficult, even in studying the direct spectral problem. Here we restrict ourselves only to the following uniqueness theorem and use another approach than in [25].

Theorem 3. *Let Condition $\mathcal{A}$ be fulfilled. Then the specification of the spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$ uniquely determines the function $H(x)$ along with the number h.*

We note that uniqueness of the solution of an inverse problem for a second-order integro-differential operator with a single discontinuity of a special type was discussed in [26].