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Infinite propagation speed and asymptotic behavior for a generalized Camassa–Holm equation with cubic nonlinearity

Wenjun Cui, Lijia Han*

Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China

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ABSTRACT

This paper is devoted to the study of the infinite propagation speed and the asymptotic behavior for a generalized Camassa–Holm equation with cubic nonlinearity. First, we get the infinite propagation speed in the sense that the corresponding solution with compactly supported initial data does not have compact support any longer in its lifespan. Then, the asymptotic behavior of the solution at infinity is investigated. Especially, we prove that the solution decays algebraically with the same exponent as that of the initial data.

1. Introduction

In this paper, we consider the Cauchy problem for the following generalized Camassa–Holm equation with cubic nonlinearity

$$\begin{cases} u_t - u_{xxt} = (1+\partial_x)(u^2u_{xx} + uu_x^2 - 2u^2u_x) & t>0, \quad x\in\mathbb{R}, \\ u(x,0) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.1}$$

Eq. (1.1) was first proposed by Novikov and it can be regarded as the negative flow of integrable quasi-linear scalar evolution equations [1]. Eq. (1.1) is integrable and possesses an infinite hierarchy of local higher symmetries [1]. In [2], Li and Yin established the local well-posedness result for (1.1) in the Besov space $B^s_{p,r}$ by applying the Littlewood–Paley theory and the transport theory. The blow-up result and the precise blow-up rate for (1.1) were also obtained in [2].

Recently, various types of Camassa–Holm equations with cubic nonlinearity have attracted much attention and interest, please read Refs. [3,4] and the references therein. One celebrated example of Camassa–Holm

* Corresponding author.
E-mail address: hljmath@ncepu.edu.cn (L. Han).

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type equations with cubic nonlinearity is the Novikov equation

$$\begin{cases}(1-\partial_x^2)u_t = 3uu_xu_{xx} + u^2u_{xxx} - 4u^2u_x, & t>0, \quad x\in\mathbb{R},\\ u(x,0)=u_0(x), & x\in\mathbb{R},\end{cases} \tag{1.2}$$

the other is the Fokas–Olver–Rosenau–Qiao (FORQ) equation

$$\begin{cases}(1-\partial_x^2)u_t = 4uu_xu_{xx} + u^2u_{xxx} + u_x^3 - 3u^2u_x - 2u_xu_{xx}^2 - u_x^2u_{xxx}, & t>0, \quad x\in\mathbb{R},\\ u(x,0)=u_0(x), & x\in\mathbb{R}.\end{cases} \tag{1.3}$$

Eq. (1.2) was discovered in [1] in a symmetry classification of nonlocal partial differential equations with quadratic or cubic nonlinearity. Eq. (1.2) has a bi-Hamiltonian structure and an infinite sequence of conserved quantities. Peakon solutions of (1.2) were studied in [5]. Other results of Eq. (1.2) were shown in [1,6–8].

Eq. (1.3) was first introduced by Fuchssteiner, Olver and Rosenau as a new generalization of integrable system by implementing a simple explicit algorithm based on the bi-Hamiltonian representation of the classically integrable system [9–11]. Eq. (1.3) could also be derived from the two dimensional Euler equations [12]. The N-soliton solution to Eq. (1.3) was obtained in [13]. The existence and dynamical stability under small perturbations of periodic peakons in energy space H^1 were obtained in [14]. Wu and Guo in [15] studied the infinite propagation speed of the solution to Eq. (1.3) under the assumption that the initial potential $m_0 = u_0 - u_{0,xx}$ did not change sign on $\mathbb{R}$ and m_0 had compact support. The traveling wave solution and persistence properties of the strong solution to Eq. (1.3) were also established in [15].

To our best knowledge, the infinite propagation speed and the asymptotic behavior of Eq. (1.1) have not been studied yet. Motivated by [15,16], our purpose in this paper is to study the infinite propagation speed and the asymptotic behavior for Eq. (1.1). Notice that [2,17]

$$(1-\partial_x)^{-1}m(x) = (1-\partial_x^2)^{-1}(1+\partial_x)m(x) = \int_x^\infty e^{x-\xi}m(\xi)d\xi,$$

so we employ $m = u - u_x$ in Eq. (1.1) and rewrite (1.1) to

$$\begin{cases}m_t + u^2m_x + uu_xm = 0, & t>0, \quad x\in\mathbb{R},\\ m(x,0)=m_0(x), & x\in\mathbb{R}.\end{cases} \tag{1.4}$$

For Eq. (1.4), we first assume initial data m_0 and u_0 have compact support, we establish the finite propagation speed for the solution $m(\cdot,t)$ to Eq. (1.4) in the sense that it retains the property of having compact support for any further time. We further assume that m_0 does not change sign on $\mathbb{R}$, we prove that the solution u, which retains the property of having compact support for any further time, is the trivial solution $u\equiv 0$. In this sense, the localized disturbance represented by u_0 propagates with an infinite speed. Although the nontrivial solution u is no longer compactly supported, we show a detailed description about the profile of the solution u in Theorem 1.3. Moreover, the asymptotic behavior at infinity of the solution u with algebraically decaying initial data is investigated in Theorem 1.4.

Now we are in a position to state the main results in each part.

Theorem 1.1. *Let $1\le p,r\le\infty$, $s\ge 2$, $m_0\in B^s_{p,r}$ and $T>0$ be the maximal existence time of the corresponding solution m to Eq. (1.4), which is guaranteed by Lemma 2.2. Assume m_0 has compact support in the interval $[\alpha_{m_0},\beta_{m_0}]$ in Eq. (1.4), then for any $t\in[0,T)$, the solution $m(x,t)$ has compact support.*

Theorem 1.2. *Let $1\le p,r\le\infty$, $s\ge 2$, $m_0\in B^s_{p,r}$, $m_0\not\equiv 0$, m_0 does not change sign on $\mathbb{R}$ and u_0 has compact support in the interval $[\alpha_{u_0},\beta_{u_0}]$. $T>0$ is the maximal existence time of the corresponding solution m to Eq. (1.4), which is guaranteed by Lemma 2.2. If the solution $u(x,t)$ with initial data $u_0(x)$ has compact support at any $t\in[0,T)$, then u is identically zero.*