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Cluster method in composites and its convergence

Vladimir Mityushev

Institute of Computer Sciences, Pedagogical University, ul. Podchorazych 2, Krakow 30-084, Poland

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Extensions of Maxwell's self-consistent approach from single- to n- inclusions problems lead to cluster methods applied to computation of the effective properties of composites. We describe applications of Maxwell's formalism to finite clusters and explain the uncertainty arising when n tends to infinity by study of the corresponding conditionally convergent series.

1. Introduction

Maxwell's self-consisting approach [1, page 365] is one of the most widely used methods for calculating the effective properties of composites. It is based on calculation of the field disturbed by a single inclusion embedded in a uniform host and further estimation of the macroscopic response via the dipole moments induced by many inclusions. A vast number of studies were initiated by this approach (see for instance [2–4] and works cited therein). Extensions of Maxwell's approach from single- to n- inclusions problems are called cluster methods. From mathematical point of view, the cluster method means extension of the constructive solution to boundary value problems for a simply connected domain to a multiply connected domain and further averaging of the obtained local fields to estimate the effective constants. It differs from cluster analysis devoted to formation of clusters and to self-assembly studied as dynamical processes [5]. In the same time, the cluster method can resolve questions posed in cluster analysis, e.g. [6].

The obtained results can be summarized as follows:

(i) Basing on the field around a finite cluster without clusters interactions one can deduce a formula for the effective conductivity only for dilute clusters.

(ii) The uncertainty arising in various self-consisting cluster methods when the number of elements n in a cluster C_n tends to infinity is analyzed by means of the conditionally convergent series discussed in [7,8].

E-mail address: mityu@up.krakow.pl.

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We consider the 2D conductivity problem with n circular inclusions solved exactly in [9] and derive the limit fromulae, as $n \to \infty$. Discussion of this simple problem justifies necessity to consider the points (i) and (ii) during manipulations in the framework of Maxwell's formalism.

2. Finite cluster

Let $z = x + iy$ denote a complex variable in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Consider non-overlapping disks $|z - a_k| < r$ $(k = 1, 2, \dots, n)$, denoted below by D_k, of conductivity σ imbedded in the host material of the normalized unit conductivity occupying the domain D, the complement of all the disks $|z - a_k| \le r$ to $\widehat{\mathbb{C}}$. The potentials $u(z)$ and $u_k(z)$ are harmonic in D and D_k, respectively, and continuously differentiable in the closures of the considered domains except at infinity where $u(z) \sim x = \mathrm{Re}\, z$. The singularity of $u(z)$ determine the external flux applied at infinity.

The perfect contact condition (transmission problem [10]) between the components is expressed by two real relations [4]

$$u_k(z) = u(z), \quad \sigma \frac{\partial u_k}{\partial \mathbf{n}}(z) = \frac{\partial u}{\partial \mathbf{n}}(z), \quad |z - a_k| = r \ (k = 1, 2, \dots, n) \tag{1}$$

where $\frac{\partial}{\partial \mathbf{n}}$ denotes the outward unit normal derivative to $|z - a_k| = r$. Introduce the contrast parameter $\varrho = \frac{\sigma - 1}{\sigma + 1}$. Two real equations (1) are reduced to the $\mathbb{R}$−linear complex condition [4]

$$\varphi(z) = \varphi_k(z) - \varrho \overline{\varphi_k(z)}, \quad |z - a_k| = r \ (k = 1, 2, \dots, n) \tag{2}$$

where $\varphi(z)$ and $\varphi_k(z)$ are analytic in D and D_k, respectively, and continuously differentiable in the closures of the considered domains except at infinity where $\varphi(z) \sim z$. The harmonic and analytic functions are related by the equalities $u(z) = \mathrm{Re}\ \varphi(z)$ in D, $u_k(z) = \frac{2}{\sigma+1} \mathrm{Re}\ \varphi_k(z)$ in D_k.

Consider the Schottky group of inversions and their compositions with respect to the circles $|z - a_k| = r$, $k = 1, 2, \dots, n$ (plus the identity element)

$$z^*_{(k)} = \frac{r^2}{\overline{z - a_k}} + a_k, \ z^*_{(k_1 k_2 \dots, k_m)} := (z^*_{(k_2 \dots, k_{m-1})})^*_{k_1}, \quad (k_{j+1} \ne k_j). \tag{3}$$

Exact solution of the considered problem for any $|\varrho| < 1$ was found in the form of the absolutely and uniformly convergent Poincaré type series [9]

$$\varphi(z) = z + \varrho \sum_{k=1}^{n} \overline{z^*_{(k)}} + \varrho^2 \sum_{k=1}^{n} \sum_{k_1 \ne k} z^*_{(k_1 k)} + \varrho^3 \sum_{k=1}^{n} \sum_{k_1 \ne k} \sum_{k_2 \ne k_1} \overline{z^*_{(k_2 k_1 k)}} + \cdots \tag{4}$$

The set of inclusions $C_n = \cup_{k=1}^{n} D_k$ forms a finite cluster on the plane. Its important characteristic is the dipole moment $M^{(n)}$ [10] equal to the coefficient of $\varphi(z)$ on z^{-1} which can be exactly written by (4). For our purposes it is sufficient to use the asymptotic formula

$$\begin{aligned} &M^{(n)} = n \varrho r^2 \mathcal{M}^{(n)}, \\ &\mathcal{M}^{(n)} = 1 - n \varrho r^2 e_2^{(n)} + n^2 \varrho^2 r^4 e_{22}^{(n)} - n^3 \varrho^2 r^6 [2 e_{33}^{(n)} + \varrho e_{222}^{(n)}] + O(r^8). \end{aligned} \tag{5}$$

Here, the multiple sums arisen from (4) are shortly written as

$$\begin{aligned} &e_p^{(n)} = \frac{1}{n^p} \sum_{k, k_1} \frac{1}{(a_k - a_{k_1})^p}, \ e_{pp}^{(n)} = \frac{1}{n^{p+1}} \sum_{k, k_1, k_2} \frac{1}{(a_k - a_{k_1})^p \overline{(a_{k_1} - a_{k_2})}^p} \\ &e_{ppp}^{(n)} = \frac{1}{n^{p+2}} \sum_{k, k_1, k_2, k_3} \frac{1}{(a_k - a_{k_1})^p \overline{(a_{k_1} - a_{k_2})}^p (a_{k_2} - a_{k_3})^p} \quad (p = 2, 3). \end{aligned} \tag{6}$$