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Multiple solutions for impulsive problems with non-autonomous perturbations[☆]

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ABSTRACT

In this article, we study the existence of multiple solutions for nonlinear impulsive problems with small non-autonomous perturbations. We show the existence of at least three distinct classical solutions by using variational methods and a three critical points theorem.

1. Introduction

The pioneering research of impulsive differential equation via variational methods was initiated by Nieto and O'Regan [1], Tian and Ge [2], and study of second order impulsive differential equation with derivative dependence ordinary differential equations via variational methods was initiated by Nieto [3].

Liu and Zhao [4], considered nonlinear boundary value problems for second order impulsive differential equations. Authors used critical point theory and variational methods to obtain at least one, two, and infinitely many classical solutions. In [5], authors established the existence of three solutions for a perturbed two point boundary value problem depending on two real parameters by variational methods. In [6], the multiplicity of solutions for a fourth-order impulsive differential equation with Dirichlet boundary conditions and two control parameters was obtained via variational methods and a three critical points theorem.

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Motivated by the above mentioned work, in this paper we consider the existence of at least three classical solutions to the impulsive boundary value problem with small non-autonomous perturbations

$$\begin{cases} -u''(t) + u(t) + p(t)u'(t) = \lambda f(t,u) + \mu g(t,u), \quad \text{a.e. } t \in [0,T], \\ \Delta u'(t_i) = I_i(u(t_i)), \quad i = 1,2,\dots,n, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\mu \geq 0$, $f, g \in C([0,T]\times\mathbb{R},\mathbb{R})$, $p \in L^\infty([0,T])$ satisfies $0 = t_0 < \frac{T}{4} \leq t_1 < t_2 < \cdots < t_n \leq \frac{3T}{4} < t_{n+1} = T$, $I_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,\dots,n$ are continuous, $\Delta u'(t_i) = u'(t_i^+) - u'(t_i^-) = \lim_{t\to t_i^+} u'(t) - \lim_{t\to t_i^-} u'(t)$.

We obtain some new existence of solutions by using variational methods combining with a three critical points theorem.

2. Preliminaries and variational structure

The following statement comes easily by the results contained in [7], which is the main tool for the proof of our result.

Theorem 2.1 *([7, Theorem 3.6]). Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that

(i) $\sup_{\Phi(x)\leq r} \Psi(x) < r\,\Psi(\bar{x})/\Phi(\bar{x})$,

(ii) *for each λ in*

$$\Lambda_r := \Big[\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}\Big],$$

the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Let $\alpha = \min_{t\in[0,T]} e^{P(t)}$, $\beta = \max_{t\in[0,T]} e^{P(t)}$, where $P(t) = -\int_0^t p(s)ds$.

We denote the Sobolev space $H := H_0^1([0,T]) = \big\{u : [0,T] \to \mathbb{R} | u$ is absolutely continuous, $u(0) = u(T) = 0$ and $u' \in L^2([0,T])\big\}$ with the inner product and the corresponding norm

$$(u,v)_H = \int_0^T e^{P(t)}(u(t)v(t) + u'(t)v'(t))dt,$$

$$\|u\|_H = \Big(\int_0^T e^{P(t)}(u(t)^2 + u'(t)^2)dt\Big)^{1/2}.$$

Obviously, H is a reflexive Banach space.

Let $H^2([0,T]) = \{u : [0,T] \to \mathbb{R} | u, u'$ are absolutely continuous, $u'' \in L^2([0,T])\}$. For $u \in H^2([0,T])$, we have that u, u' are both absolutely continuous, and $u'' \in L^2([0,T])$. Hence $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ for any $t \in (0,T)$. If $u \in H([0,T])$, we have that u is absolutely continuous, and $u' \in L^2(0,T;\mathbb{R})$, thus the one side derivatives $u'(t^+)$, $u'(t^-)$ may not exist, which leads to the impulsive effects.

So by a classical solution to problem (1.1) we mean a function $u \in C[0,T]$ satisfying (1.1) such that $u_i = u_{|(t_i,t_{i+1})} \in H^2(t_i,t_{i+1})$ and $u'(t_i^-), u'(t_i^+)$ exist for every $i = 1,2,\dots,n$ and verify the impulsive and