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Finite time stability of fractional delay differential equations ☆

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ABSTRACT

In this paper, we firstly introduce a concept of delayed Mittag-Leffler type matrix function, an extension of Mittag-Leffler matrix function for linear fractional ODEs, which help us to seek explicit formula of solutions to fractional delay differential equations by using the variation of constants method. Secondly, we present the finite time stability results by virtue of delayed Mittag-Leffler type matrix. Finally, an example is given to illustrate our theoretical results.

1. Introduction

In recent decades, existence and finite time stability (FTS) problems of integer order and fractional order delay differential equations have been studied by using methods of linear matrix inequality, Lyapunov functions, and Gronwall's integral inequality, see for example [1–11].

Recently, Khusainov and Shuklin [12] give a new concept, delayed exponential matrix function, which is used as a representation of solution of a linear delay differential equation. Next, it is worth to mention that Diblík and Khusainov [13] transfer this idea to represent the solution of discrete delayed system by constructing discrete matrix delayed exponential. For more recent results about delay equations based on delayed exponential matrix, one can refer to [14–17] and the references therein.

In this paper, we apply a new method to study FTS of the following linear fractional delay differential equations:

$$\begin{cases} ({}^cD^{\alpha}_{0+}y)(x) = By(x-\tau),\ y(x) \in \mathbb{R}^n,\ x \in J := [0,T],\ \tau > 0, \\ y(x) = \varphi(x),\ \ -\tau \le x \le 0, \end{cases} \tag{1}$$

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where ${}^cD^{\alpha}_{0+}y(\cdot)$ is the Caputo derivative of order $\alpha \in (0,1)$ and of lower limit zero, $B \in \mathbb{R}^{n\times n}$ denotes constant matrix, $T = k^*\tau$ for a fixed $k^* \in \Lambda := \{1, 2, \ldots\}$, τ is a fixed delay time, and $\varphi(\cdot)$ is an arbitrary continuously differentiable vector function, i.e., $\varphi \in C^1_\tau := C^1([-\tau, 0], \mathbb{R}^n)$.

We adopt similar idea from [12] and introduce delayed Mittag-Leffler type matrix function, which will be applied to seek a representation of solution of (1). Some new sufficient conditions to guarantee FTS of (1) are established by virtue of delayed Mittag-Leffler type matrix.

2. Preliminaries

Throughout the paper, we denote $\|y\| = \sum_{i=1}^{n}|y_i|$ and $\|A\| = \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|$, which are the Euclidean vector norm and matrix norm, respectively; y_i and a_{ij} are the elements of the vector y and the matrix A, respectively. Denote by $C(J, \mathbb{R}^n)$ the Banach space of vector-value continuous function from $J \to \mathbb{R}^n$ endowed with the norm $\|x\|_C = \max_{t\in J}\|x(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. We introduce a space $C^1(J, \mathbb{R}^n) = \{x \in C(J, \mathbb{R}^n) : x' \in C(J, \mathbb{R}^n)\}$. In addition, we note $\|\varphi\|_C = \max_{\theta\in[-\tau,0]}\|\varphi(\theta)\|$.

We recall definitions of Caputo fractional derivative and finite time stability.

Definition 2.1 (*See [18]*)**.** The Caputo derivative of order $0 < \alpha < 1$ for a function $f : [0,\infty) \to \mathbb{R}$ can be written as $({}^cD^{\alpha}_{0+}y)(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{y'(t)}{(x-t)^{\alpha}}dt,\ x > 0$.

Definition 2.2 (*See [1]*)**.** System given by (1) is finite time stable with respect to $\{0, J, \tau, \delta, \beta\}$ if and only if $\|\varphi\|_C < \delta$ implies $\|y(x)\| < \beta,\ \forall\ x \in J$, where $\varphi(x),\ -\tau \le x \le 0$ is the initial time of observation, δ, β are real positive numbers and $\delta < \beta$.

Next, we introduce a concept of delayed Mittag-Leffler type matrix function, which is an analogy of delayed exponential matrix e^{Bt}_τ in [12].

Definition 2.3. Delayed Mittag-Leffler type matrix function $\mathbb{E}^{Bx^\alpha}_\tau : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined by

$$\mathbb{E}^{Bx^\alpha}_\tau = \begin{cases} \Theta, & -\infty < x < -\tau, \\ I, & -\tau \le x \le 0, \\ I + B\dfrac{x^\alpha}{\Gamma(\alpha+1)} + B^2\dfrac{(x-\tau)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + B^k\dfrac{(x-(k-1)\tau)^{k\alpha}}{\Gamma(k\alpha+1)}, & \\ \qquad (k-1)\tau \le x \le k\tau,\ k \in \Lambda, & \end{cases} \tag{2}$$

where B is an $n\times n$ constant matrix, Θ and I are the zero and identity matrices, respectively.

Lemma 2.4. *For any $x \in [(k-1)\tau, k\tau]$ and $k \in \Lambda$, we have*

$$\left\|\mathbb{E}^{Bx^\alpha}_\tau\right\| \le \mathbb{E}_\alpha(\|B\|x^\alpha),$$

where $\mathbb{E}_\alpha(\cdot)$ is the Mittag-Leffler function defined by $\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \alpha > 0, z \in \mathbb{R}$.

Proof. By the formula of (2), one has

$$\begin{aligned}\|\mathbb{E}^{Bx^\alpha}_\tau\| &\le 1 + \|B\|\frac{x^\alpha}{\Gamma(\alpha+1)} + \|B\|^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \|B\|^k\frac{x^{k\alpha}}{\Gamma(k\alpha+1)} \\ &\le \sum_{k=0}^{\infty}\frac{(\|B\|x^\alpha)^k}{\Gamma(k\alpha+1)} = \mathbb{E}_\alpha(\|B\|x^\alpha).\end{aligned}$$

The proof is completed. □