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Schwartz duality of the Dirac delta function for the Chebyshev collocation approximation to the fractional advection equation

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ABSTRACT

Singular source terms represented by the Dirac delta function are found in various applications modeling natural problems. Solutions to differential equations perturbed by such singular source terms have jump discontinuity and their high order numerical approximations suffer from the Gibbs phenomenon. We use the Schwartz duality to approximate the Dirac delta function existent in fractional differential equations. The singular source term is approximated by the fractional derivative of the Heaviside function. We provide a Chebyshev spectral collocation method for solving the fractional advection equation with the singular source term and show that the Schwartz duality yields the consistent formulation resulting in vanishing Gibbs phenomenon. The numerical results show that the proposed approximation of the Dirac delta function is efficient and accurate, particularly for linear problems.

1. Introduction

We are interested in solving numerically the following fractional advection equation with the singular source term

$$\begin{cases} u_t + {}_aD_x^\alpha u(x,t) = \delta(x-1-a), & x \in [a, a+2], \quad t > 0, \\ u(x,0) = f(x), & x \in [a, a+2], \\ u(a,t) = g(t), & t > 0, \end{cases} \tag{1.1}$$

where $\delta(x)$ is the Dirac delta function and ${}_aD_x^\alpha$ is the fractional derivative of order α. With the definition of fractional integral of order α ($\alpha > 0$)

$${}_aD_x^{-\alpha} v(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} v(\xi) d\xi, \quad x > a, \tag{1.2}$$

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the fractional derivative can be defined as a natural extension of (1.2). The Riemann–Liouville fractional derivative is

$$ {}_aD_x^\alpha v(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-\alpha-1} v(\xi)d\xi, \quad x > a, \quad \alpha \in (n-1, n), \tag{1.3} $$

and the Caputo fractional derivative, an alternative closely related to the Riemann–Liouville fractional derivative, is

$$ {}_a^CD_x^\alpha v(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n v(\xi)}{d\xi^n} d\xi, \quad x > a, \quad \alpha \in (n-1, n). \tag{1.4} $$

Since the Caputo derivative is more convenient to specify suitable initial and boundary conditions of fractional differential equations [1], we focus on the Caputo fractional derivative. Various numerical methods have been developed for solving fractional partial differential equations (FPDEs), including finite difference methods [2,3], spectral methods [4–8] and discontinuous Galerkin methods [9]. The main objective of this paper is to provide a consistent Chebyshev spectral collocation method for solving the fractional advection equation with the singular source term which is approximated by the Schwartz duality on the collocation points.

The paper is organized as follows. In Section 2, we derive the Chebyshev fractional differentiation matrix. The fractional Schwartz duality for the approximation of the Dirac delta function is proposed in Section 3. In Section 4, we present the analytical solution of our model problem followed by some numerical experiments in Section 5. In Section 6, a concluding remark is provided.

2. Chebyshev spectral collocation method for fractional derivative

Here we derive the Chebyshev pseudo-spectral fractional differentiation matrix based on the three term recurrence relation. Let $T_j(x)$ be the jth order Chebyshev polynomial of the first kind defined on $[-1, 1]$, satisfying the three-term recurrence relation with $T_0(x) = 1$ and $T_1(x) = x$

$$ T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x), \quad j \geq 1. \tag{2.5} $$

A variation of the above relation is also given by the derivatives as below

$$ 2T_j(x) = \frac{1}{j+1} \frac{d}{dx} T_{j+1}(x) - \frac{1}{j-1} \frac{d}{dx} T_{j-1}(x), \quad j \geq 2. \tag{2.6} $$

For $0 < \alpha < 1$, the fractional derivative of the Chebyshev polynomial of degree j is given by

$$ T_j^{(\alpha)} := {}_{-1}D_x^\alpha T_j(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-1}^x (x-s)^{-\alpha} \frac{dT_j(s)}{ds} ds. \tag{2.7} $$

In order to derive the recurrence relation of $T_j^{(\alpha)}$, we need the following lemma.

Lemma 2.1. *Let $K_j = \frac{1}{\Gamma(1-\alpha)} \int_{-1}^x (x-s)^{-\alpha} T_j(s)ds$, then the following equality is satisfied for $j \geq 3$*

$$ K_j = \frac{2jx}{j+1-\alpha} K_{j-1} - \frac{j(j-3+\alpha)}{(j-2)(j+1-\alpha)} K_{j-2} + \frac{2(-1)^{j+1}(x+1)^{1-\alpha}}{(j+1-\alpha)(j-2)\Gamma(1-\alpha)}. \tag{2.8} $$

Proof. We can prove the lemma by applying the recurrence (2.5) and (2.6). That is,

$$ \begin{aligned} K_j &= \frac{1}{\Gamma(1-\alpha)} \int_{-1}^x (x-s)^{-\alpha} [2sT_{j-1}(s) - T_{j-2}(s)]ds \\ &= 2xK_{j-1} - K_{j-2} - \frac{2}{\Gamma(1-\alpha)} \int_{-1}^x (x-s)^{1-\alpha} T_{j-1}(s)ds. \end{aligned} \tag{2.9} $$