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Well-posedness of a general higher order model in nonlinear acoustics

Barbara Kaltenbacher

Institute of Mathematics, Alpen-Adria Universität Klagenfurt, Universitätsstr. 65–67, 9020 Klagenfurt, Austria

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ABSTRACT

In this paper we prove well-posedness and exponential stability for a general model of nonlinear acoustics. The key steps of the proof are a multiplicative splitting of the differential operator into a heat and a strongly damped wave part plus some lower order remainder, as well as derivation of an appropriate spectral bound.

1. Introduction

Driven by the increasing number of applications (e.g. in high intensity (focused) ultrasound HIFU), the field of nonlinear acoustics and its mathematical modeling has been highly active recently, see, e.g., [1–8] and further references in [9].

We here prove well-posedness of the general higher order model of nonlinear acoustics proposed by Brunnhuber and Jordan in [1], additionally using the linear approximation $\Delta\psi \approx c0\psi_{tt}$, as allowed under the weakly non-linear scheme [10]

$$-\tfrac{\nu^2}{\mathrm{Pr}}\gamma\varLambda\Delta^2\partial_t\psi - c_0^2\tfrac{\nu}{\mathrm{Pr}}\Delta^2\psi + \left(\varLambda\nu + \gamma\tfrac{\nu}{\mathrm{Pr}}\right)\Delta\partial_{tt}\psi + c_0^2\Delta\partial_t\psi - \partial_{ttt}\psi = \partial_{tt}\left(\frac{\gamma-1}{2c_0^2}(\partial_t\psi)^2 + \sigma\left|\nabla\psi\right|^2\right) \tag{1.1}$$

where $\sigma \in \{0, 1\}$ and all coefficients are positive, i.e.,

$$\partial_{ttt}\psi + \mathrm{A}\Delta^2\psi + \mathrm{B}\Delta^2\partial_t\psi - \mathrm{C}\Delta\partial_t\psi - \mathrm{D}\Delta\partial_{tt}\psi = -\partial_{tt}\left(\frac{k}{2}(\partial_t\psi)^2 + \sigma\left|\nabla\psi\right|^2\right) \tag{1.2}$$

with $\mathrm{A} = c_0^2\frac{\nu}{\mathrm{Pr}}$, $\mathrm{B} = \frac{\nu^2}{\mathrm{Pr}}\gamma\varLambda$, $\mathrm{C} = c_0^2$, $\mathrm{D} = \varLambda\nu + \gamma\frac{\nu}{\mathrm{Pr}}$, $k = \frac{\gamma-1}{c_0^2}$. Note that (1.1) is just [1, eq. (4)], with the right hand side written in such a way that it contains both the Westervelt type and the Kuznetsov type nonlinearity via choosing $\sigma \in \{0, 1\}$.

E-mail address: barbara.Kaltenbacher@aau.at.

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We consider (1.1) on some C^4 domain $\Omega \subseteq \mathbb{R}^n$ and impose homogeneous Dirichlet boundary conditions on ψ and $\Delta\psi$ and initial conditions $(\psi, \partial_t\psi, \partial_{tt}\psi)(0) = (\psi_0, \psi_1, \psi_2)$.

2. Well-posedness and exponential decay

The key idea here is to rewrite the differential operator on the left hand side of (1.2) as a perturbation of the one analyzed in [11] (see also [12,13])

$$\partial_{ttt} + \mathrm{A}\Delta^2 + \mathrm{B}\Delta^2\partial_t - \mathrm{C}\Delta\partial_t - \mathrm{D}\Delta\partial_{tt} = (\partial_t - a\Delta)(\partial_{tt} - c^2\Delta - b\Delta\partial_t) - r\Delta\partial_t \tag{2.1}$$

with $a = \frac{\mathrm{D}\pm\sqrt{\mathrm{D}^2-4\mathrm{B}}}{2}$, $b = \mathrm{D} - a$, $c^2 = \frac{\mathrm{A}}{a}$, $r = \mathrm{C} - c^2$, i.e.,

$$a = \Lambda\nu, \qquad b = \gamma\tfrac{\nu}{\mathrm{Pr}}, \qquad c^2 = c_0^2\frac{1}{\Lambda\mathrm{Pr}}, \qquad r = c_0^2\left(1 - \frac{1}{\Lambda\mathrm{Pr}}\right)$$

or

$$a = \gamma\tfrac{\nu}{\mathrm{Pr}}, \qquad b = \Lambda\nu, \qquad c^2 = \frac{c_0^2}{\gamma}, \qquad r = c_0^2\left(1 - \frac{1}{\gamma}\right) > 0 \quad \text{by } \gamma > 1.$$

We choose the second version, since nonnegativity of the coefficient r will turn out to be crucial.

Indeed, well-posedness of the linearized problem

$$\begin{aligned} &(\partial_t - a\Delta)(\partial_{tt} - c^2\Delta - b\Delta\partial_t)u - r\Delta\partial_t u = f \\ &u = \Delta u = 0 \quad \text{on } \partial\Omega \\ &u(0) = u_0, \qquad \partial_t u(0) = u_1, \qquad \partial_{tt}u(0) = u_2 \end{aligned} \tag{2.2}$$

with $a, b, c^2 > 0$, $r \geq 0$ follows from the results in [11] via a simple perturbation argument, as well as an appropriate estimate of the spectral bound.

For this purpose, we rewrite (2.2) as an abstract ODE

$$U_t(t) = AU(t) + F(t) \quad t \in (0,T), \qquad U(0) = U_0$$

with

$$U = \begin{pmatrix} u \\ \partial_t u \\ \partial_{tt}u - c^2\Delta u - b\Delta\partial_t u \end{pmatrix}, \qquad U_0 = \begin{pmatrix} u_0 \\ u_1 \\ u_2 - c^2\Delta u_0 - b\Delta u_1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}$$

$$A = A_0 + B \quad A_0 = \begin{pmatrix} 0 & I & 0 \\ -c^2\mathcal{A} & -b\mathcal{A} & I \\ 0 & 0 & -a\mathcal{A} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -r\mathcal{A} & 0 \end{pmatrix}, \tag{2.3}$$

$$H = \mathcal{D}(\mathcal{A}^2) \times \mathcal{D}(\mathcal{A}) \times \mathcal{H}, \qquad \mathcal{D}(A) = \mathcal{D}(\mathcal{A}^2) \times \mathcal{D}(\mathcal{A}^2) \times \mathcal{D}(\mathcal{A})$$

where $\mathcal{A} = -\Delta$ with homogeneous Dirichlet boundary conditions, $\mathcal{H} = L_2(\Omega)$, $\mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega)$. As shown in [11, Proposition 3.1], A_0 generates a uniformly exponentially stable analytic semigroup and obviously $B \in L(H)$. Thus we can immediately conclude the following result from [11] and the perturbation theorem for analytic semigroups, see, e.g., [14, Theorem III.2.10].

Proposition 1. *The operator $A : \mathcal{D}(A) \to H$ defined in (2.3) generates an analytic semigroup.*

To conclude uniform exponential stability of A, we prove that for $r \geq 0$, the spectral bound of A is strictly negative. To this end, we first of all consider the resolvent equation.