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Weak solutions for the stationary Schrödinger equation and its application☆

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ABSTRACT

In this paper, we prove Carleman's formula for weak solutions of the stationary Schrödinger equation in a cylinder. As an application of it, the integral representation of solutions of the stationary Schrödinger equation is also obtained.

1. Introduction and main results

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$, in $\mathbf{R}^n$ $(n \geq 2)$ which are related to Cartesian coordinates $(x_1, x_2, \ldots, \theta_{n-1}, y)$ by $y = r\cos\theta_1$. For positive functions h_1 and h_2, we say that $h_1 \lesssim h_2$ if $h_1 \leq ch_2$ for some constant $c > 0$. If $h_1 \lesssim h_2$ and $h_2 \lesssim h_1$, then we say that $h_1 \approx h_2$.

Let Δ_n be the Laplace operator and Ω be a bounded domain in $\mathbf{R}^{n-1}$ with smooth boundary $\partial\Omega$. Consider the Dirichlet problem $(\Delta_{n-1} + \lambda)\varphi = 0$ on Ω and $\varphi = 0$ on $\partial\Omega$ (see [1, p. 41]). We denote the least positive eigenvalue of this boundary value problem by λ and the normalized positive eigenfunction corresponding to λ by φ, $\int_\Omega \varphi^2(X)d\Omega = 1$, where $X \in \Omega$ and $d\Omega$ is the $(n-1)$-dimensional volume element.

The set $\Omega \times \mathbf{R} = \{P = (X, y) \in \mathbf{R}^n; X \in \Omega, y \in \mathbf{R}\}$ in $\mathbf{R}^n$ is simply denoted by $T_n(\Omega)$. We call it a cylinder (see [2]). In the following, we denote the sets $\Omega \times I$ and $\partial\Omega \times I$ with an interval I on $\mathbf{R}$ by $T_n(\Omega; I)$ and $S_n(\Omega; I)$ respectively. Hence $S_n(\Omega; \mathbf{R})$ denoted simply by $S_n(\Omega)$ is $\partial T_n(\Omega)$.

Let $\mathscr{A}_\Omega$ denote the class of nonnegative radial potentials $a(P)$ (i.e. $0 \leq a(P) = a(r)$ for $P = (X, y) = (r, \Theta) \in T_n(\Omega)$) such that $a \in L^b_{\rm loc}(T_n(\Omega))$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

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For the identical operator I and the potential $a(P) \in \mathscr{A}_\Omega$, we define the stationary Schrödinger operator by $SSE_a = -\Delta_n + a(P)I$, where SSE_a can be extended in the usual way from the space $C_0^\infty(T_n(\Omega))$ to an essentially self-adjoint operator on $L^2(T_n(\Omega))$ (see [3, Ch. 11] for more details). Furthermore SSE_a has a Green–Sch function $\mathcal{G}_\Omega^a(P,Q)$ (associated with the stationary Schrödinger operator SSE_a). Here $\mathcal{G}_\Omega^a(P,Q)$ is positive on $T_n(\Omega)$ and its inner normal derivative $\partial\mathcal{G}_\Omega^a(P,Q)/\partial n_Q$ is non-negative, where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $T_n(\Omega)$. We write $1/c_n \partial\mathcal{G}_\Omega^a(P,Q)/\partial n_Q$ by $\mathcal{PI}_\Omega^a(P,Q)$, which is called the Poisson–Sch kernel (associated with the stationary Schrödinger operator SSE_a) on $T_n(\Omega)$. Here, $c_2 = 2$ and $c_n = (n-2)w_n$ when $n \geq 3$, where w_n is the surface area of the unit sphere in $\mathbf{R}^n$. The Poisson–Sch integral $\mathcal{PI}_\Omega^a[g](P)$ of g (associated with the stationary Schrödinger operator SSE_a) on $T_n(\Omega)$ is defined as follows $\mathcal{PI}_\Omega^a[g](P) = \int_{S_n(\Omega)} \mathcal{PI}_\Omega^a(P,Q)g(Q)d\sigma_Q$, where g is a continuous function on $S_n(\Omega)$ and $d\sigma_Q$ is the surface area element on $S_n(\Omega)$.

Let $\mathscr{B}_\Omega$ be the class of the potential $a(P) \in \mathscr{A}_\Omega (P = (X,y) = (r,\Theta) \in T_n(\Omega))$ such that $\lim_{r\to\infty} r^2 a(r) = \kappa_0 \in [0,\infty)$ and $r^{-1}|r^2 a(r) - \kappa_0| \in L(1,\infty)$.

We denote by $SpH_a(\Omega)$ the class of all weak solutions of $SSE_a u(P) \geq 0$ for any $P \in T_n(\Omega)$, which are continuous when $a \in \mathscr{B}_\Omega$ (see e.g. [4]). In the rest of paper, we will always assume that $a \in \mathscr{B}_\Omega$. If $u(P) \in SpH_a(\Omega)$ and $-u(P) \in SpH_a(\Omega)$, then $u(P)$ is the solution of $SSE_a u(P) = 0$ for any $P \in T_n(\Omega)$. The class of all solutions of it is denoted by $H_a(\Omega)$.

An important role will be played by the solutions of the ordinary differential equation

$$-\Pi''(y) + (\lambda + a(|y|))\,\Pi(y) = 0 \quad (-\infty < y < +\infty), \tag{1.1}$$

which has two specially linearly independent positive solutions $V(y)$ and $W(y)$ on $\mathbf{R}$ such that $V(y)$ is nondecreasing and $W(y)$ is nonincreasing as $y \to \pm\infty$ (see [3, Ch. 11, Appendix C] for more details). The solutions $V(y)$ and $W(y)$ of Eq. (1.1) have the asymptotics $V(y) \approx \exp(\sqrt{\lambda}y)$ and $W(y) \approx \exp(-\sqrt{\lambda}y)$, as $y \to \pm\infty$ (see [3,5]). We remark that both $V(y)\varphi(X) \in H_a(\Omega)$ and $W(y)\varphi(X) \in H_a(\Omega)$ vanish continuously on $S_n(\Omega)$.

Let $u(P) \in SpH_a(\Omega)$, we use the stand notations $u^+ = \max\{u,0\}$ and $u^- = -\min\{u,0\}$. The integral $\int_\Omega u(P)\varphi(X)d\Omega$ of $u(P)$ is denoted by $N(u)(y)$ when it exists, where $P = (X,y)$. The finite or infinite limits $\lim_{y\to+\infty} N(u)(y)/V(y)$ and $\lim_{y\to-\infty} N(u)(y)/W(y)$ are denoted by $\mathscr{U}_V(u)$ and $\mathscr{V}_W(u)$ respectively, when they exist.

About Carleman's formula for harmonic functions in a half space and a smooth cone, we refer the reader to the papers by Zhang et al. (see [6]) and Ronkin (see [7]) respectively. Our main aim in this paper is to prove Carleman's formula for weak solutions of the stationary Schrödinger equation in a truncated cylinder. An application of it will be also given later.

Theorem 1.1. *Let $0 < r < R < +\infty$ and define*

$$\Psi(y) = V(y)\left(\frac{W(y)}{V(y)} - \frac{W(R)}{V(R)}\right)$$

where $r < |y| < R$. If $u(X,y) \in SpH_a(\Omega)$, then we have

$$\int_{T_n(\Omega,(-R,-r))} \mathfrak{T}(X,y) SSE_a u(X,y) dw$$
$$= \frac{\chi'(R)}{V(R)} N(u)(-R) + \int_{S_n(\Omega,(-R,-r))} u(X',y')\,\Psi(-y')\frac{\partial\varphi(X')}{\partial n_{X'}} d\sigma_Q + d_1(-r) + \frac{W(R)}{V(R)} d_2(-r) \tag{1.2}$$

and

$$\int_{T_n(\Omega,(r,R))} \mathfrak{T}(X,y) SSE_a u(X,y) dw$$
$$= \frac{\chi'(R)}{V(R)} N(u)(R) + \int_{S_n(\Omega,(r,R))} u(X',y')\,\Psi(y')\frac{\partial\varphi(X')}{\partial n_{X'}} d\sigma_Q + d_1(r) + \frac{W(R)}{V(R)} d_2(r) \tag{1.3}$$